Geometric Entropy for Self-Gravitating Systems in Alternative Theories of Gravity

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Abstract: conserved quantities are usually computed in covariant theories by implementing variational techniques based on N"other’s theorem. As such, they depend on the variational principle, while a correct physical interpretation would prefer to associate conserved quantities directly to solutions.

We shall present here some explicit examples of dynamically equivalent theories, with different conservation laws which by various mechanisms eventually define the same conserved quantities for the corresponding solutions.

1. Introduction

The problem of conservation laws and conserved quantities in General Relativity (GR) is a long standing one. Many prescriptions have been proposed, some covariant \cite{1,2,3,4} some based on pseudo-tensors \cite{5}, some based on variational calculus \cite{2}, some directly on solutions \cite{6}.

Recently the covariant variational prescriptions have been considerably extended. They now cover many situations with different asymptotic behaviours \cite{7,8}, for various variational principles \cite{9}. The variational techniques have proved to be effective in shading light onto the relations between different \textit{ad hoc} prescriptions \cite{10} and in providing a framework connecting the mathematical prescriptions to our physical interpretation.

The relation between covariant conservation laws and entropy of singular solutions have been ultimately established \cite{11,12}. Given a certain set of solutions and thermodynamic potentials the entropy is defined as the quantity which satisfies a variational equation implementing the first principle of thermodynamics. Although the microscopic origin of such an entropy will probably remain obscure until a reliable framework for quantum gravity will be established, the macroscopic setting based on classical variational calculus has proven to provide a unifying language to investigating relations between different approaches and able to cover many physically relevant situations.

One of the fundamental problems of covariant variational conservation techniques (see \cite{9,13}) is that conservation laws are technically related to variational principles, while our physical intuition would prefer to associate conserved quantities directly to solutions.

In GR many different variational formulations are known to provide different descriptions of the same physical situations; they are found to be more or less physically equivalent, though using different fundamental fields and/or (dynamically equivalent) actions and field equations \cite{14,15}.

Usually, different frameworks identify different field equations, each selecting a solution space. Dynamical equivalence may be generically described by mapping the solution spaces onto each other establishing a correspondence among solutions that represent the same physical situations in different frameworks. The weak point of such a situation is that \textit{a priori} it is not clear if or why
conserved quantities associated with solutions in different frameworks should be preserved by such correspondences. To our knowledge no general theory is still known about this important issue.

We shall present here some detailed examples taken from physically very relevant situations, in which we shall explicitly identify how, despite different conservation laws are produced in dynamically equivalent frameworks, the corresponding conserved quantities at least are not affected by the same ambiguity.

2. Relativistic theories

The modern approach to the study of physical field theories (see [16]) strongly relies on variational principles: the physical systems are described by sections of a fiber bundle (called configuration bundle) $C$ over a spacetime manifold $M$ of dimension $\dim(M) = m$; let us denote fibered coordinates on $C$ by $(x^\mu, y^i)$. We refer to [16] for notation and further details. Here $\mu$ runs from 1 to $m = \dim(M)$ and $i$ runs from 1 to $\dim(C) - m$.

Field equations are generated by variational calculus from a Lagrangian $L = \mathcal{L}(j^k y) ds$. Here $ds$ is the standard local basis for $m$-forms over $M$ induced by coordinates $x^\mu$ (i.e. the local “volume element”). The Lagrangian $L$ can be viewed as a horizontal $m$-form over a suitable jet prolongation $J^k C$ of the configuration bundle, or equivalently as a bundle morphism $L : J^k C \to A_m(M)$ into the bundle $A_m(M)$ of $m$-forms over $M$, providing a functional of the section, namely its action functional. Usually, one has $k = 1$ or $k = 2$.

One can associate to any global Lagrangian two morphisms: $\mathbb{E}(L)$, called Euler-Lagrange morphism, and $\mathbb{F}(L)$, called Poincaré-Cartan morphism, such that the first variational formula holds true

$$\langle \delta L | j^k X \rangle = \langle \mathbb{E}(L) | X \rangle + \text{Div}(\mathbb{F}(L) | j^{k-1} X)$$

for any variation, namely for any vertical vector field $X = \delta y^i \partial_i$ on $C$. Here $\text{Div}$ denotes the divergence operator acting on $(m-1)$-horizontal forms on $J^k C$.

If one restricts to (gauge-)covariant theories there are two classes of configuration bundles available from the physical viewpoint: natural bundles, which are associated to a higher order frame bundle $L^k(M)$ for some $k$ (though in many cases the ordinary frame bundle obtained for $k = 1$ is enough and $k = 2$ covers also linear connections); and gauge natural bundles, which are associated to (a gauge natural prolongation of) a principal bundle $P$ with gauge group $G$.

The former case is related to relativistic theories which are covariant with respect to spacetime diffeomorphisms (i.e. the group $\text{Diff}(M)$), while the latter has to do with relativistic theories that are covariant with respect to some wider group of gauge transformations.

The situation in natural theories is represented by the following diagram

$$
\begin{array}{cccc}
\text{Aut}(L^k(M)) & \longrightarrow & \text{Aut}(C) & \\
\downarrow & & & \\
L^k(\cdot) & \longrightarrow & C & \\
\downarrow & & \downarrow & \\
\text{Diff}(M) & \longrightarrow & M & \\
\end{array}
$$

(2.2)
The configuration bundle is associated to the frame bundle $L^k(M)$, any spacetime diffeomorphism $f$ can be lifted to the frame bundle and then induces an automorphism $\hat{f}$ on the configuration bundle $C$. This means that sections of $C$, representing fields, do transform “covariantly” with respect to spacetimes diffeomorphisms. Not for all physical fields such an action can be defined in an observer independent manner; for tensor fields, metrics, and their Levi-Civita connections such actions are known and standard, while for electromagnetic (as well as Yang-Mills) fields such an action cannot be defined unless the gauge is fixed.

For this reason such more general physical situations are caught by the following diagram, representing the situation in so-called gauge natural theories

$$\begin{array}{ccc}
\text{Aut}(P) & \longrightarrow & \text{Aut}(C) \\
\downarrow \pi_* & & \downarrow \pi_* \\
\text{Diff}(M) & \longrightarrow & M \\
\end{array}$$

Notice that the mapping $\pi_*$ goes in the opposite direction with respect to $L^k(\cdot)$ preventing one from defining “covariantly” the action of spacetime diffeomorphisms onto fields in $C$. In this case, gauge transformations in $\text{Aut}(P)$ do act instead on fields, as well as project on spacetime diffeomorphisms. One can also define pure gauge transformation $\text{Aut}_V(P)$ as the subset of gauge transformations which project on $\text{id}_M \in \text{Diff}(M)$ producing the following group exact sequence

$$\begin{array}{ccccc}
\mathbb{I} & \longrightarrow & \text{Aut}_V(P) & \longrightarrow & \text{Aut}(P) & \longrightarrow & \pi_* \text{Diff}(M) & \longrightarrow & \mathbb{I} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
& & & & 1 & & & & 1
\end{array}$$

Spacetime diffeomorphisms do not (and cannot) act on fields directly, though gauge transformations encompass both pure gauge transformations and spacetime diffeomorphisms (natural theories are recovered by just setting the gauge group to be trivial, $G = \{1\}$).

Any principal connection $\omega$ on $P$ provides however a splitting of the sequence and induces a dynamical action of spacetime diffeomorphisms on fields. Such an action is dynamical since it depends on $\omega$ which is in fact a dynamical field.

In the case of General Relativity (in vacuum or with matter) gauge natural field theories (as well as natural theories which are here considered as a trivial case of gauge natural theories) have been used as a general framework. This general setting is especially suitable to deal with purely metric, metric-affine (also known as Palatini), tetrads and tetrad-affine frameworks as well as spinor matter. In all these cases fields are sections of a gauge-natural bundle, although certain combinations (such as the Lagrangian density) may be tensorial.

3. Geometric framework for conserved quantities

Natural and gauge natural field theories are also the suitable framework to deal with conservation laws. Using the properties of the first variational formula (2.1) we can write down a version of the Nöther theorem.

A gauge trasformation is a Lagrangian symmetry iff it leaves the Lagrangian $L$ (and consequently its action, its field equations and its solution space) invariant. Given a one-parameter
family of Lagrangian symmetries its infinitesimal generator \( \Xi \) is a right–invariant vector field on \( P \) projecting on a spacetime vector field \( \xi \) on \( M \). Moreover, the invariance of the Lagrangian is encoded into the covariance identity

\[
p_i \mathcal{L}_{\Xi y^i} + p_i^\mu d_\mu (\mathcal{L}_{\Xi y^i}) + \ldots = d_\mu (\xi^\mu \mathcal{L})
\]  

(3.1)

where \((p_i, p_i^\mu, \ldots)\) are the Lagrangian momenta with respect to \((y^i, y_i^\mu, \ldots)\), respectively, and \(\mathcal{L}_{\Xi y^i}\) denotes the Lie derivative of fields along the gauge transformation.

The Lie derivative of a configuration field is a (generalized) vertical vector field on the configuration bundle \( C \) (just as the deformations \( X = \delta y^i \partial_i \) are) so that covariance identity (3.1) can be written intrinsically as

\[
< \delta L \mid \mathcal{L}_{\Xi y} > = \text{Div}(\xi J L)
\]  

(3.2)

Now using the first variation formula (2.1) one can easily recast the covariance identity as

\[
\text{Div}(\mathcal{E}(L, \Xi)) = \mathcal{W}(L, \Xi)
\]  

(3.3)

where we set

\[
\mathcal{E}(L, \Xi) = < \mathcal{F} \mid j^{k-1} \mathcal{L}_{\Xi y} > - i_\xi L
\]

\[
\mathcal{W}(L, \Xi) = - < \mathcal{E} \mid \mathcal{L}_{\Xi y} >
\]

The identity (3.3) is a weak conservation law since it says that the quantity \(\mathcal{E}(L, \Xi)\) is a closed form (hence conserved) when calculated on-shell, i.e. along solutions of field equations.

Let us here notice that since by construction the Nöther current \(\mathcal{E}(L, \Xi)\) is a global \((m - 1)\)-form, the covariant exterior differential (with respect to a torsionless connection) is equal to the exterior differential and the covariant conservation laws is in fact a continuity equation.

Thanks to a further covariant integration by parts one obtains

\[
\mathcal{W}(L, \Xi) = \mathcal{B}(L, \Xi) + \text{Div}\mathcal{E}(L, \Xi)
\]

\[
\mathcal{E}(L, \Xi) = \mathcal{E}(L, \Xi) + \text{Div}\mathcal{U}(L, \Xi)
\]

The term \(\mathcal{B}(L, \Xi)\) is named Bianchi morphism and it generalizes the Bianchi identities, i.e. it is vanishing off-shell, on any configuration; the quantity \(\mathcal{E}(L, \Xi)\) is vanishing only on-shell and is called reduced current, while finally the term \(\mathcal{U}(L, \Xi)\) is called superpotential.

The importance of the superpotential is in the definition of covariant conserved quantities: let \(\Omega \subset M\) be a closed spacetime \((m - 2)\)-region and \(\mathcal{U}(L, \Xi)\) the superpotential and let \(\sigma\) be a solution of field equations; then we can define the conserved quantity \(Q_\Omega\) to be

\[
Q_\Omega(\Xi, \sigma) = \int_\Omega \mathcal{U}(L, \Xi, \sigma) = \int_\Omega (\mathcal{J}^{k-2} \sigma)^* \mathcal{U}(L, \Xi)
\]

We have to remark that the superpotential is a primitive for the Nöther current; once this is computed on-shell is not only closed but also exact, regardless of the topology of spacetime.

Physically speaking, when the base manifold \(M\) has dimension 4, the calculations of conserved quantities are analogues to the definition of electric charge by means of Gauss’s theorem.
4. Nöther approach to Entropy

Using the framework shortly summarized in the previous sections we can define a quantity associated to solutions describing self–gravitant system that can be called entropy; see [11], [12].

Let us consider (see [9], [13]) a black hole solution of GR, for example the so–called Kerr solution: it describes a gravitational system with axial-symmetry by means of the metric

\[ g_K = -\left(1 - \frac{2Mr}{\rho^2}\right)dt \otimes dt + \frac{\rho^2}{\Delta} dr \otimes dr + \rho^2 d\theta \otimes d\theta + \left(r^2 + a^2 + \frac{2Mar^2}{\rho^2} \sin^2 \theta \right) \sin^2 \theta d\phi \otimes d\phi - \frac{4Mar}{\rho^2} \sin^2 \theta d\phi \otimes dt \]

where

\[ \Delta = r^2 - 2Mr + a^2 \quad \rho^2 = r^2 + a^2 \cos^2 \theta \]

The superpotential for this theory based on the purely metric Hilbert Lagrangian of GR

\[ L_H = R \sqrt{g} ds \]

has the expression

\[ \mathcal{U}(L_H, \xi) = \nabla^{[\xi^\alpha]} \sqrt{g} ds_{\alpha\beta} \]

being \( ds_{\alpha\beta} = \partial_{[\alpha} \sqrt{g} ds_{\beta]} \) and it is known to reproduce for the Kerr solution (and in fact for a larger family of solutions) the correct value for the angular momentum \( (J = ma) \) but just one half of the expected value of the mass \( (M = m) \) upon integration at spatial infinity, i.e.

\[ M = 2 \int_{\infty} \mathcal{U}(L_H, \partial_t, g_K) \]
\[ J = \int_{\infty} \mathcal{U}(L_H, \partial_\phi, g_K) \]

This is known as the anomalous factor problem (see [17]) and we shall see one reasonable and coherent way to solve this in the next Section.

It is known that a relation between the parameters of the black hole solutions can be written down so that one can obtain a formula that looks like the first principle of thermodynamics

\[ \delta M = \frac{1}{T} \delta S + \Omega \delta J + \ldots \]

where the physical quantities \( \Omega, T \) are obtained by means of suitable physical considerations, and they are interpreted as thermodynamical potentials; in this case, as the angular velocity and the temperature, respectively. We shall interpret accordingly the quantity \( S \) as the entropy of the self-gravitating system.
5. Augmented formalism

In order to solve the anomalous factor problem we define the variation of conserved quantity (see [3], [12]) as

$$\delta_X Q(L, \xi) = \int \delta_X \mathcal{U}(L, \xi) - i_\xi <\mathbb{F}|X>$$

that has proven to be effective in all known examples as Kerr [12], BTZ [7], Taub-Bold [8] and so on.

Let us introduce (see [2]) a vacuum field $\overline{\mathcal{F}}$; then the correction term along a family of solutions generated by $X = \delta y^i \partial_i$ is given by

$$-i_\xi <\mathbb{F}|X> = -\overline{\mathcal{F}}^{[y^j]_{\xi}} \delta y^i ds_{\mu\nu}$$

In order to obtain a Lagrangian with the correct quantities we can use the freedom to add any divergence to the Lagrangian: we consider then the augmented Lagrangian on $J^2(C \times M \mathcal{C})$

$$l (j^2y, j^2\overline{y}) = L (j^1y) - L (j^1\overline{y}) + \text{Div} (jy, j\overline{y}),$$

where $jy$ and $j\overline{y}$ denote the jet prolongations to some suitable order; the superpotential of this Lagrangian is given by

$$\mathcal{U}(l, \Xi) = \mathcal{U}(L, \Xi) - \mathcal{U}(\overline{\mathcal{L}}, \Xi) + \alpha [\mu, \xi] \delta y^i ds_{\mu\nu}$$

where we have denoted by $\mathcal{L}$ the Lagrangian dependent on $\overline{y}$.

The pure divergence term $\alpha$ can be fixed so that the superpotential of $l()$ reproduces (5.1) when evaluated for ($y = \overline{y} + \delta y, \overline{y}$).

This can be easily done by fixing $\alpha$ so that it obeys the following condition:

$$\left. \frac{d}{ds} \alpha (jy_s, j\overline{y}) \right|_{s=0} = - <\mathbb{F}(\overline{\mathcal{L}})|X> \quad \text{and with } \alpha (y_0, j\overline{y}) = 0$$

where $y_s$ is a 1-parameter family of solutions generated by integration along $X$, starting from the initial condition $y_0 = \overline{y}$.

In the case of the standard theory of General Relativity the Poincaré-Cartan morphism is

$$<\mathbb{F}|j^1X> = \sqrt{g} g^{\alpha\beta} \delta u^\lambda_{\alpha\beta} ds_\lambda$$

where $u^\lambda_{\alpha\beta} = \Gamma^\lambda_{\alpha\beta} - \delta^\lambda_\alpha \Gamma^\alpha_{\beta}, \Gamma^\lambda_{\alpha\beta}$ denoting the coefficients of the Levi-Civita connection of $g$, and $\Gamma^\beta = \Gamma^\alpha_{\alpha\beta}$ denotes the trace of the connection, being $ds_\lambda = \partial_\lambda ds$. Defining $w^\lambda_{\alpha\beta} = u^\lambda_{\alpha\beta} - \overline{w}^\lambda_{\alpha\beta}$ we obtain the augmented Lagrangian as

$$l (jg, j\overline{y}) = R \sqrt{g} - \overline{R} \sqrt{\overline{g}} - d_\lambda \left( \sqrt{\overline{g}} \overline{g}^{\alpha\beta} w^\lambda_{\alpha\beta} \right)$$

The conserved quantities computed from this Lagrangian have been proved to correct the anomalous factor problem in all asymptotically flat solutions [12], to provide good results also for solutions which are not asymptotically flat [2], [7], and to apply to solutions with singularities of much stranger structure than ordinary black holes, [8].
6. Alternative theories of gravity

Let us now analyze the case of alternative theories of gravity obtained from the Lagrangian

\[ L(g, \Gamma) = \sqrt{g} f(R) \, ds \]

(see [9], [14], [15]), where \( f \) is a generic analytical function of one real variable, \( g \) and \( \Gamma \) are a (Lorentzian) metric and a linear symmetric connection considered as independent fields and, as such, obeying the fields equations

\[
\begin{align*}
\left\{ 
\begin{array}{l}
f'(R) R_{\alpha\beta} - \frac{1}{2} f(R) g_{\alpha\beta} = 0 \\
\nabla_\lambda \left( \sqrt{g} f'(R) g^{\alpha\beta} \right) = 0
\end{array}
\right.
\end{align*}
\]

where \( R = R(g, \Gamma) = g^{\mu\nu} R_{\mu\nu}(\Gamma) \) and \( R_{\mu\nu}(\Gamma) \) is the Ricci tensor of the linear connection \( \Gamma \). Here \( f'(R) = \frac{df}{dR} \) and \( \nabla \) is the covariant derivative along \( \Gamma \). We also assume \( m \neq 2 \).

To obtain the augmented Lagrangian for these theories we start with the calculation of Poincaré-Cartan morphism, that gives:

\[
\langle F | X \rangle = \sqrt{g} f'(R) g^{\alpha\beta} \delta u^\lambda_{\alpha\beta} \, ds_\lambda
\]

Accordingly, the augmented Lagrangian is (with obvious notation)

\[
l = \sqrt{g} f(R) - \sqrt{\tilde{g}} f(\tilde{R}) - \text{Div} \left( \sqrt{g} f'(R) g^{\alpha\beta} w^\lambda_{\alpha\beta} \right)
\]

where \((\tilde{g}, \tilde{\Gamma})\) are a reference couple of fields.

Tracing the first field equation we obtain

\[
f'(R) R - \frac{m}{2} f(R) = 0 \quad (6.1)
\]

that is, for a given \( f \), an algebraic equation for \( R \) which generically has a finite number of simple zeros (we exclude the case \( f(R) = R^m/2 \), i.e. \( f(R) \neq R^2 \) in dimension 4, a case in which this is identically satisfied). Let \( R = \rho \) be one of such simple zeros and let us replace it into the field equations, so that the second one reads now as

\[
\nabla_\lambda \left( \sqrt{g} f'(\rho) g^{\alpha\beta} \right) = 0 \quad \Rightarrow \quad \nabla_\lambda g^{\alpha\beta} = 0
\]

and the first equation reads as

\[
f'(\rho) \left( R_{\alpha\beta} - \frac{1}{2} \rho g_{\alpha\beta} \right) = \frac{2 - m}{4} f(\rho) g_{\alpha\beta}
\]

that can be recasted as

\[
R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = \Lambda g_{\alpha\beta}
\]

setting \( \Lambda = - \frac{m-2}{2m} \rho \). Notice that \( m \neq 2 \) by hypothesis and that assuming \( \rho \) to be a simple zero of (6.1) entails that \( f'(\rho) \neq 0 \) too.

The superpotential of the augmented Lagrangian is thence

\[
U(L, \xi) = \sqrt{g} \left( f'(R) \nabla^\mu \xi^\nu + \frac{1}{2} f''(R) \nabla_\lambda R \, g^{\lambda[\mu} \xi^{\nu]} \right) \, ds_{\mu\nu} + \sqrt{\tilde{g}} \left( f'(\tilde{R}) \nabla^\mu \xi^\nu + \frac{1}{2} f''(\tilde{R}) \nabla_\lambda \tilde{R} \, g^{\lambda[\mu} \xi^{\nu]} \right) \, ds_{\mu\nu} + \sqrt{\tilde{g}} \left( f'(\rho) \tilde{g}^{\alpha\beta} w^\lambda_{\alpha\beta} \xi^\nu \right) \, ds_{\mu\nu}
\]

\[
= f'(\rho) \left[ \sqrt{g} \nabla^\mu \xi^\nu - \sqrt{\tilde{g}} \nabla^\mu \xi^\nu + \sqrt{\tilde{g}} \left( \tilde{g}^{\alpha\beta} w^\lambda_{\alpha\beta} \xi^\nu \right) \right] \, ds_{\mu\nu}
\]
Here we have used the fact that on-shell one has $\nabla_\lambda R = 0$. Superpotentials agree on-shell with the augmented superpotential of Hilbert (with cosmological constant) up to a normalisation constant.

For example the Kerr-AdS metric

$$g_{AdS} = -\frac{\Delta_r}{\rho^2} \left[ dt - \frac{a \sin^2 \theta}{\Xi} \right]^2 + \frac{\rho^2}{\Delta_r} dr^2 + \frac{\rho^2}{\Delta_\theta} d\theta^2 + \frac{\Delta_\theta \sin^2 \theta}{\rho^2} \left[ a dt - \frac{r^2 + a^2}{\Xi} d\phi \right]^2$$

where $\rho^2 = r^2 + a^2 \cos^2 \theta$, $\Xi = 1 - \frac{a^2}{r^2}$, $\Delta_r = (r^2 + a^2)(1 + \frac{r^2}{l^2}) - 2mr$, $\Delta_\theta = 1 - \frac{a^2}{r^2} \cos^2 \theta$. This metric generalizes Kerr black hole solution $g_K$ to an asymptotically anti-deSitter spacetime.

It is a solution of standard General Relativity with cosmological constant $\Lambda = -\frac{3}{l^2}$ but it is also a solution of any alternative theory with an $f$ having a simple zero in $\rho$ so conserved quantities, and in particular the entropy, of $g_{AdS}$ is the same when computed in the two frameworks.

Non-linear theories of the Palatini form $f(R)$ provide an example of theories dynamically equivalent to Einstein GR (with a suitable cosmological constant); in this case the superpotential agrees on-shell with the superpotential of Einstein GR (up to a normalization constant).

7. The Holst’s formulation

Another interesting case is the Holst Lagrangian (see [18], [19]), used in the framework of Loop Quantum Gravity and there defined as

$$L_h(e^a, \omega^{ab}) = R^{ab} \wedge e^c \wedge e^d \epsilon_{abcd} + \frac{2}{\gamma} R^{ab} \wedge e_a \wedge e_b = L_1 + \frac{2}{\gamma} L_2$$

where $L_1$ is the standard Lagrangian for General Relativity in the frame-affine formalism. Here $e^a$ is a (co)tetrad field, $\omega^{ab}$ is a spacetime spin connection and $R^{ab}$ its curvature 2-form (see [16], [18], [20]). The real parameter $\gamma \neq 0$ is known as the Immirzi parameter while $e_{abcd}$ is standard the Ricci antisymmetric tensor density.

The augmented Lagrangian in this case is defined using as correction terms

$$\left\{ \begin{align*}
\alpha_1 &= (\omega^{ab} - \mathcal{W}^{ab}) \wedge e^c \wedge e^d \epsilon_{abcd} \\
\alpha_2 &= (\omega^{ab} - \mathcal{W}^{ab}) \wedge e_a \wedge e_b
\end{align*} \right.$$

We assume boundary conditions at the boundary surface $e = \Xi$ but $de \neq d\Xi$.

The superpotentials, using the definition of Kosmann lift $\xi_{(v)}^{ab} = e^a \epsilon^{b\beta} \nabla_\beta \xi^a$ to lift the symmetry generator, are $U_i = K_i - \hat{K}_i - i\xi_i$ where

$$K_1 = e^a \epsilon_{a} e^{cd} \epsilon_{abcd} \epsilon^{\mu\nu\rho\sigma} d s_{\mu\nu}$$

$$K_2 = e_{\mu\nu} e_{\alpha\beta} s_{(v)}^{ab} \epsilon^{\mu\nu\rho\sigma} d s_{\mu\nu}$$

$$\alpha_1 = \xi^\lambda (\omega^{ab} - \mathcal{W}^{ab}) e_a e^c \epsilon_{abcd} \epsilon^{\mu\nu\rho\sigma} d s_{\mu\lambda}$$

$$\alpha_2 = 4\sqrt{g} g^{ab} w_{\alpha\beta} \xi^\lambda d s_{\mu\lambda} + \Delta^{\mu\lambda} d s_{\mu\lambda}$$
\[ \alpha_2 = \xi^\lambda \left( \omega^a_{\mu} - \overline{\omega}^a_{\mu} \right) e^{\nu\rho} e_{\lambda\sigma} e^{\mu\nu\rho\sigma} d s_{\mu\lambda} \]
\[ = \xi^\lambda \left( e_{\lambda\sigma} d_\nu e^{b\sigma} - e_{\lambda\sigma} d_\nu e^{b\lambda} \right) e^{\mu\nu\rho\sigma} d s_{\mu\lambda} \]

where we have used the on-shell relation between frames and connections
\[ \omega^a_{\mu} = e^a_{\alpha} \left( \Gamma^\alpha_{\beta\mu} e^{b\beta} + d_\mu e^{b\alpha} \right) \]
as well as field equations and boundary conditions. Here we also set \( d s_{\mu\rho} = \partial_\rho d s_{\mu\nu} \)

The effective calculation is carried out on the boundary surface where \( e = \overline{e} \) but \( d e \neq d \overline{e} \) so that we eventually obtain
\[ \Delta^\mu\lambda = 4 \sqrt{g} \left( e^a_\nu d_\nu e^{b\mu} - e^a_\nu d_\nu e^{b\nu} - \overline{e}^a_\nu d_\nu \overline{e}^{b\mu} + \overline{e}^a_\nu d_\nu \overline{e}^{b\nu} \right) \xi^\lambda \]
\[ = 4d_\nu \left[ \sqrt{g} \xi^\lambda \left( \overline{e}^a_\nu u_b e^{b\mu} - \overline{e}^a_\nu \overline{e}^{b\mu} \right) \right] \]
i.e. a pure divergence term which does not affect the value of conserved quantities. An analogous result is obtained for \( \alpha_2 \), that can be written as
\[ \alpha_2 = d_\nu \left( \xi^\lambda \overline{e}^a_\sigma e_{b\sigma} e^{b\mu\nu\sigma} \right) \]
so that the superpotential is the same of the augmented Lagrangian in the metric formalism.

Here we have an explicit example of two dynamically equivalent theories with different superpotentials (and conservation laws). The difference reduces however to a divergence on-shell so that conserved quantities are eventually unaffected.

8. Conclusions and perspective

We have here analyzed a few examples of field theories that are dynamically equivalent to GR. In the first case conservation laws were directly unaffected, while in the second case superpotentials have been found to be different. However, the difference of superpotentials reduced on-shell to a pure divergence leaving the corresponding conserved quantities to be uneffected as well.

This result was somehow expected in view of our physical intuition, as well as the result presented in [6]. However, general results about the superpotentials and the corresponding conserved quantities of dynamically equivalent theories are still extremely scarce and deserve further investigations, as a fully general theory is to our knowledge still lacking.

In particular, dynamically equivalent theories are known to be produced by suitable Lagendre transformations; see [21]. Further investigations will be thence devoted to try to understand in general the behaviour of conservation laws with respect to such generalized Legendre transformations.
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