TWO CURVES, ONE PRICE

Pricing & Hedging Interest Rate Derivatives Using Different Yield Curves For Discounting and Forwarding

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Summary

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   - Single-Curve Pricing & Hedging Interest-Rate Derivatives
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2. **Double-Curve Framework, No Arbitrage and Basis Adjustment**
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   - Double-Curve Pricing Procedure
   - No Arbitrage Revisited and Basis Adjustment
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1: Context & Market Practices: *Single-Curve Pricing & Hedging IRD*

Pre credit-crunch market practice for pricing & hedging interest rate derivatives:

- select ***one*** finite set of the most convenient (e.g. liquid) vanilla interest rate instruments traded in real time on the market with increasing maturities; for instance, a very common choice in the EUR market is a combination of short-term EUR deposit, medium-term Futures on Euribor3M and medium-long-term swaps on Euribor6M;
- build ***one*** yield curve using the selected instruments plus a set of bootstrapping rules (e.g. pillars, priorities, interpolation, etc.);
- compute ***on the same curve*** forward rates, cashflows, discount factors and work out the prices by summing up the discounted cashflows;
- compute the delta sensitivity and hedge the resulting delta risk using the suggested amounts (hedge ratios) of the ***same*** set of vanillas.

1: Context & Market Practices:
Market Evolution

15:20 30SEP08 ICAP UK69580 ICAPEUROBASIS

EUR Basis Swaps (as 2 Swaps)

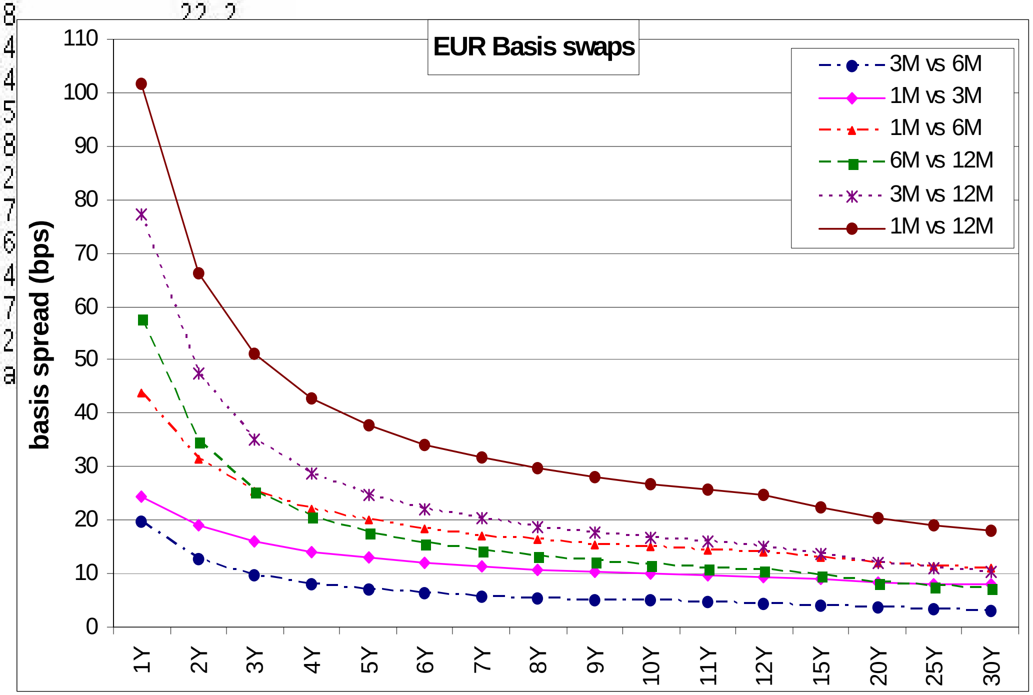
For Further Details Please Call David Shepherd on +44 (0)207 532 3530
These are indicative mids priced out of a spot starting date
All prices are Euribor vs Euribor
For other Forward Start 3X6 Basis Please See <ICAPEUROBASIS2>

	3M vs 6M	1M vs 3M	1M vs 6M	6M vs 12M	3M vs 12M
1YR	19.7	24.4	43.9	57.7	77.2
2YR	12.6	18.9	31.5	34.8	47.4
3YR	9.6	15.9	25.4	25.5	35.1
4YR	8.0	14.1	22.1	20.7	28.7
5YR	7.0	12.9	19.9	17.8	24.8
6YR	6.3	12.0	18.4	15.8	22.2
7YR	5.8	11.4	17.2	14.4	
8YR	5.5	10.8	16.3	13.4	
9YR	5.1	10.4	15.5	12.5	
10YR	4.9	10.0	14.9	11.8	
11YR	4.7	9.7	14.4	11.2	
12YR	4.5	9.4	13.9	10.7	
15YR	4.0	8.9	12.9	9.6	
20YR	3.6	8.3	11.9	8.4	
25YR	3.3	8.0	11.4	7.7	
30YR	3.1	7.9	11.0	7.2	

ICAP Global Index <ICAP> Forthcoming cha

Basis swaps as 2 swaps:

1. $Euribor3M_T$ vs R_T^{3M}
2. $Euribor6M_T$ vs R_T^{6M}
3. $Basis_T^{3M6M} = R_T^{3M} - R_T^{6M}$



Credit crunch and liquidity crisis since Aug. 2007 have ruled out the standard single-curve market practice of pricing and hedging IRD

1: Context & Market Practices: *Single-Curve Pricing & Hedging IRD*

Post credit-crunch market practice for pricing & hedging interest rate derivatives:

- build ***one discounting curve*** using the preferred procedure;
- select ***multiple separated*** sets of vanilla interest rate instruments traded in real time on the market with increasing maturities, each set ***homogeneous*** in the underlying rate (typically with 1M, 3M, 6M, 12M tenors);
- build ***multiple forwarding curves*** using the selected instruments plus their bootstrapping rules;
- compute ***on each forwarding curve*** the forward rates and the corresponding cashflows relevant for pricing derivatives on the ***same*** underlying;
- compute the corresponding discount factors using the **discounting curve** and work out prices by summing up the discounted cashflows;
- compute the delta sensitivity and hedge the resulting delta risk using the suggested amounts (hedge ratios) of the ***corresponding*** set of vanillas.

2: Double-Curve Framework: *General Assumptions*

1. There exist **two different interest rate markets** M_x, $x = \{d,f\}$ characterized by the **same currency** and by two distinct **bank accounts** B_x and **yield curves**

$$C_x := \{T \rightarrow P_x(t_0, T), T \geq t_0\},$$

2. The usual **no arbitrage relation** $P_x(t, T_2) = P_x(t, T_1) \times P_x(t, T_1, T_2)$ holds in each interest rate market M_x.
3. Simple compounded **forward rates** are defined as usual for $t \leq T_1 < T_2$

$$P_x(t, T_1, T_2) = \frac{P_x(t, T_2)}{P_x(t, T_1)} = \frac{1}{1 + F_x(t; T_1, T_2)\,\tau_x(t, T_1, T_2)},$$

4. **FRA** pricing under $Q_x^{T_2}$ forward measure associated to numeraire $P_x(t, T_2)$

$$\begin{aligned}\mathbf{FRA}_x(t; T_1, T_2, K) &= P_x(t, T_2)\,\tau_x(T_1, T_2)\left\{\mathrm{E}_t^{Q_x^{T_2}}\left[L_x(T_1, T_2)\right] - K\right\} \\ &= P_x(t, T_2)\,\tau_x(T_1, T_2)\left[F_x(t; T_1, T_2) - K\right],\end{aligned}$$

2: Double-Curve Framework: *Pricing Procedure*

1. assume $\mathcal{C}_d$ as the **discounting curve** and $\mathcal{C}_f$ as the **forwarding curve**;
2. calculate any relevant spot/forward rate **on the forwarding curve** $\mathcal{C}_f$ as

$$F_f(t;T_{i-1},T_i) = \frac{P_f(t,T_{i-1}) - P_f(t,T_i)}{\tau_f(T_{i-1},T_i)P_f(t,T_i)}, \quad t \le T_{i-1} < T_i,$$

3. calculate cashflows c_i, $i = 1,\ldots,n$, as expectations of the *i-th* coupon payoff π_i with respect to the **discounting T_i - forward measure** $Q_d^{T_i}$

$$c_i := c(t,T_i,\pi_i) = E_t^{Q_d^{T_i}}[\pi_i];$$

4. calculate the price π at time *t* by discounting each cashflow c_i using the corresponding discount factor $P_d(t,T_i)$ obtained from the **discounting curve** $\mathcal{C}_d$ and summing up,

$$\pi(t,\boldsymbol{T}) = \sum_{i=1}^{n} P(t,T_i)\,E_t^{Q_d^{T_i}}[\pi_i];$$

5. Price **FRAs** as

$$\mathbf{FRA}(t;T_1,T_2,K) = P_d(t,T_2)\,\tau_d(T_1,T_2)\left\{\mathrm{E}_t^{Q_d^{T_2}}\left[F_f(T_1;T_1,T_2)\right] - K\right\}$$

2: Double-Curve Framework: *No Arbitrage Revisited and Basis Adjustment*

The main consequence of the previous assumptions is that **standard single-curve no arbitrage relations are broken up**:

$$P_d(t,T_1,T_2) = \frac{1}{1+F_d(t;T_1,T_2)\,\tau_d(t,T_1,T_2)} \neq \frac{1}{1+F_f(t;T_1,T_2)\,\tau_f(t,T_1,T_2)}.$$

We are thus induced to postulate a **generalized double-curve no arbitrage relation**

$$P_d(t,T_1,T_2) = \frac{1}{1+F_d(t;T_1,T_2)\,\tau_d(t,T_1,T_2)} = \frac{1}{1+F_f(t;T_1,T_2)\,BA_{fd}(t;T_1,T_2)\,\tau_f(t,T_1,T_2)},$$

or the equivalent **transformation rule for forward rates**

$$F_d(t;T_1,T_2) = F_f(t;T_1,T_2)\,BA_{fd}(t;T_1,T_2).$$

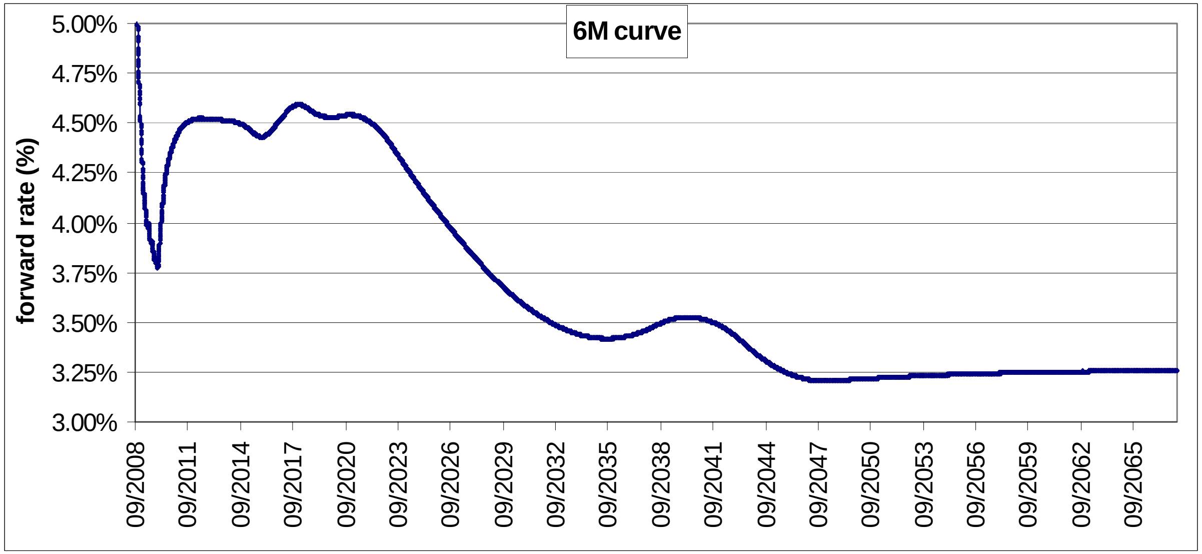
We obtain the following static expression for the **(forward) basis adjustment**

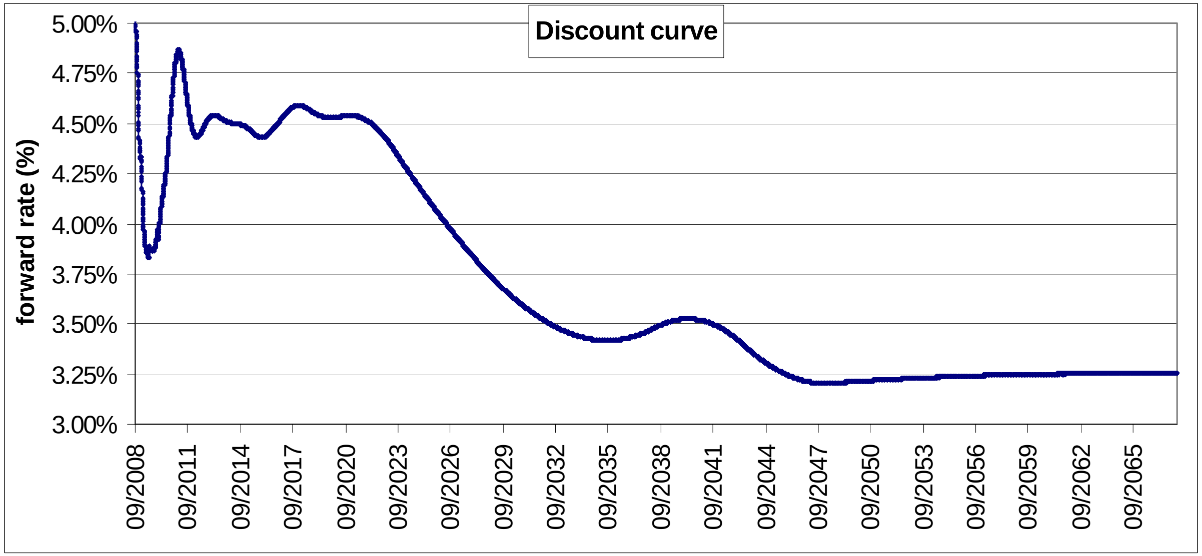
$$BA_{fd}(t;T_1,T_2) := \frac{F_d(t;T_1,T_2)}{F_f(t;T_1,T_2)} = \frac{\tau_f(T_1,T_2)}{\tau_d(T_1,T_2)}\,\frac{P_f(t,T_2)}{P_d(t,T_2)}\,\frac{P_d(t,T_1)-P_d(t,T_2)}{P_f(t,T_1)-P_f(t,T_2)}.$$

(Forward) basis adjustment can also be defined additively: we have

$$BA'_{fd}(t;T_1,T_2) = F_d(t;T_1,T_2) - F_f(t;T_1,T_2) := F_f(t;T_1,T_2)\left[BA_{fd}(t;T_1,T_2)-1\right].$$

2: Double-Curve Framework: *Forward Curves*



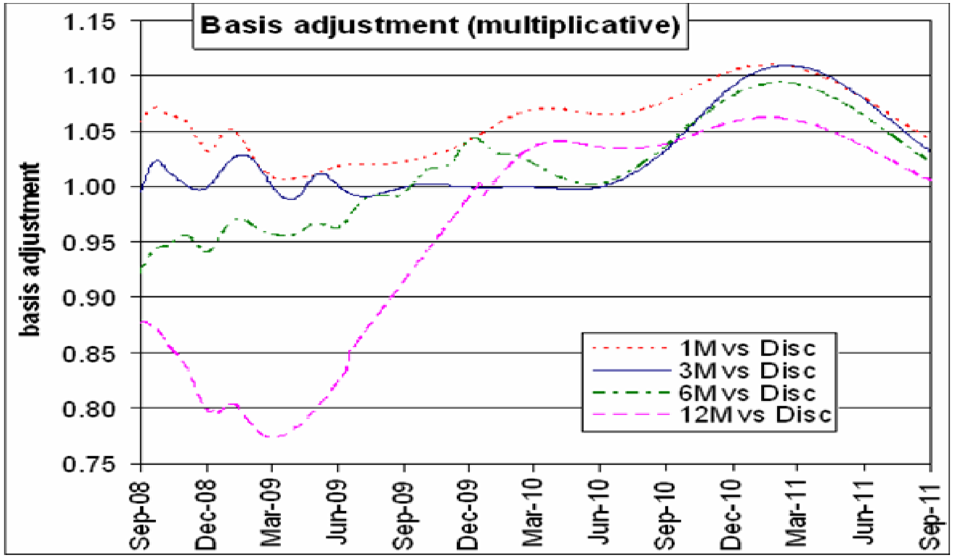


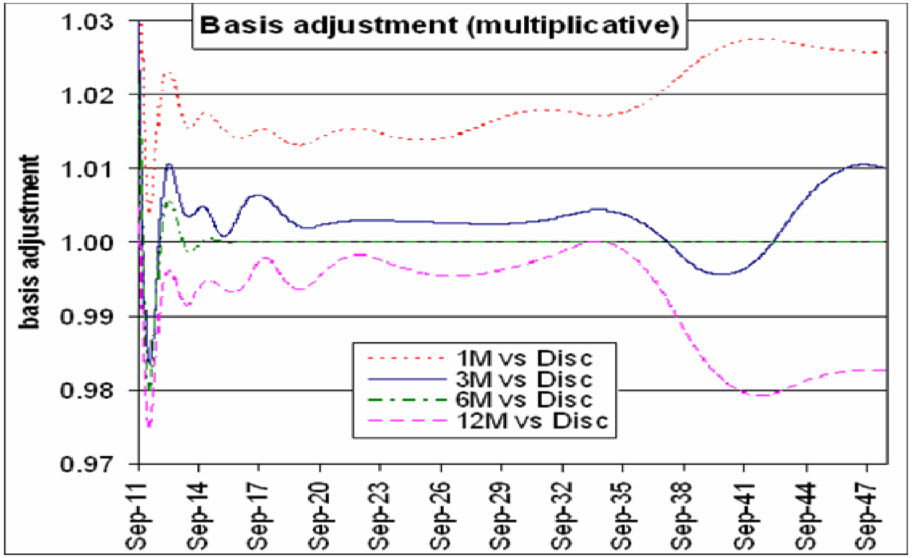
Forward curves: plots of 6M-tenor fwd rates $F_d\,(t_0;t,t+6M,act\,/\,360)$ *t* daily sampled and t_0= 15 Sep. 2008.

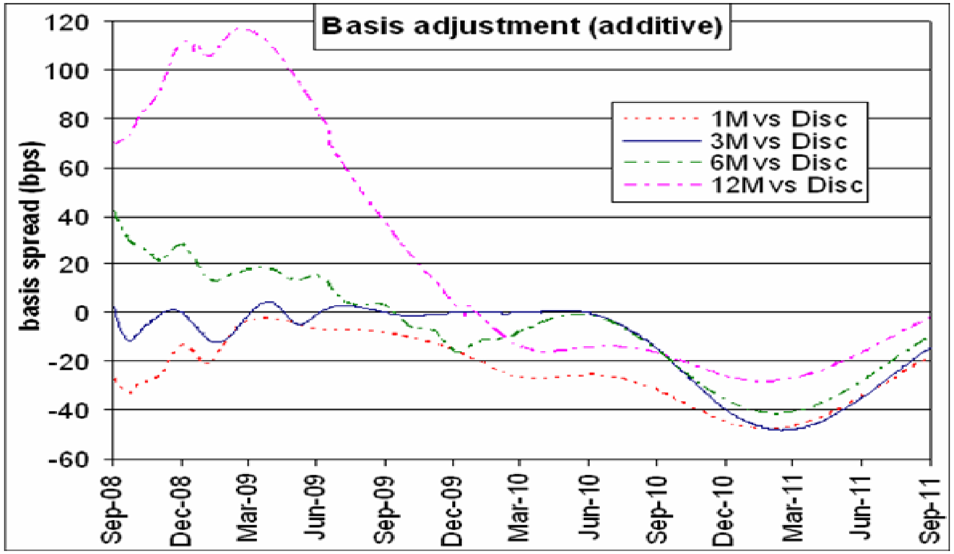
The effects of the market stress are clearly visible in the crazy roller-coaster look-up of the curves. The same pattern is observed also in the 1M, 3M, 12M curves.

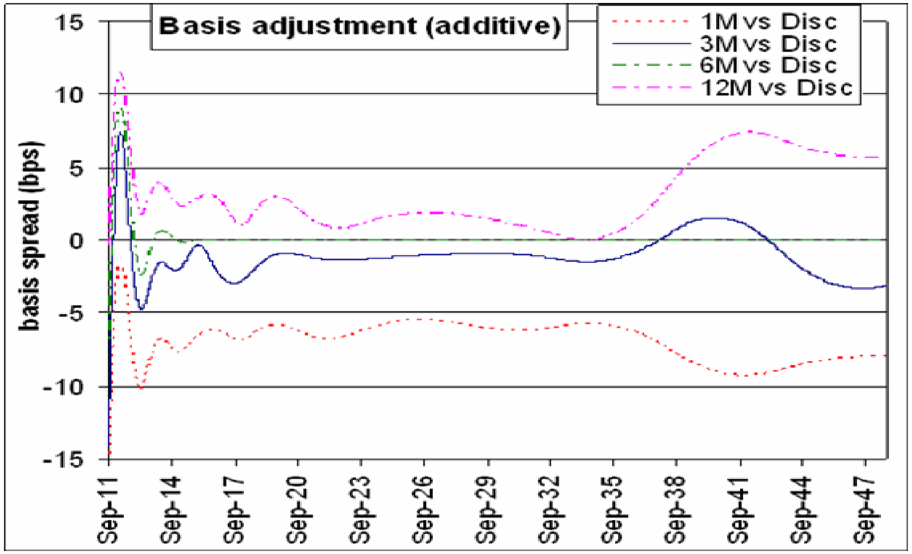
Numerical results from QuantLib (*www.quantlib.org*)

2: Double-Curve Framework:
Basis Adjustment Curves

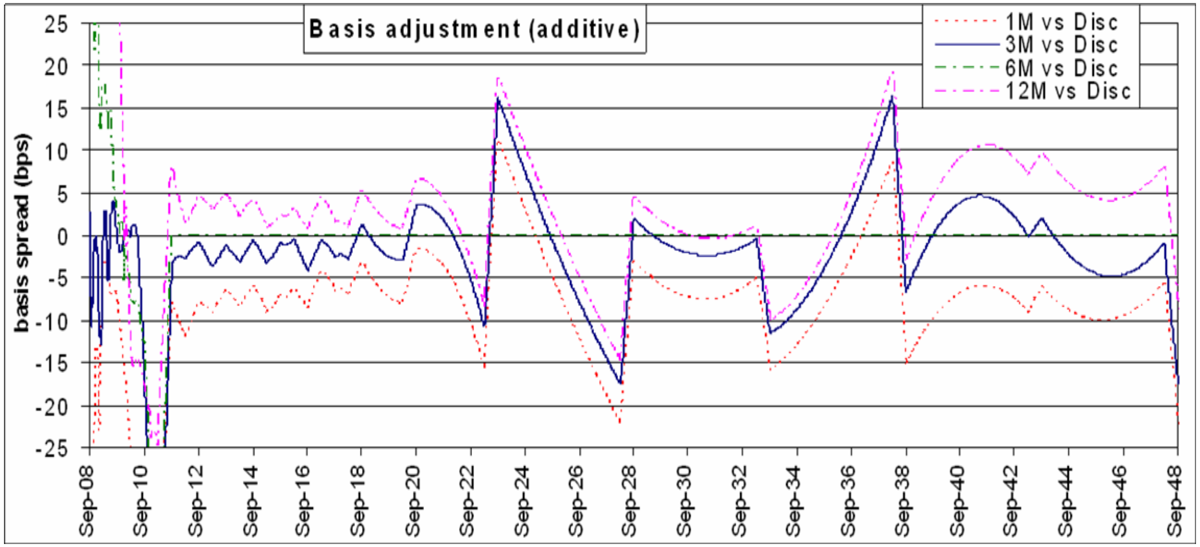


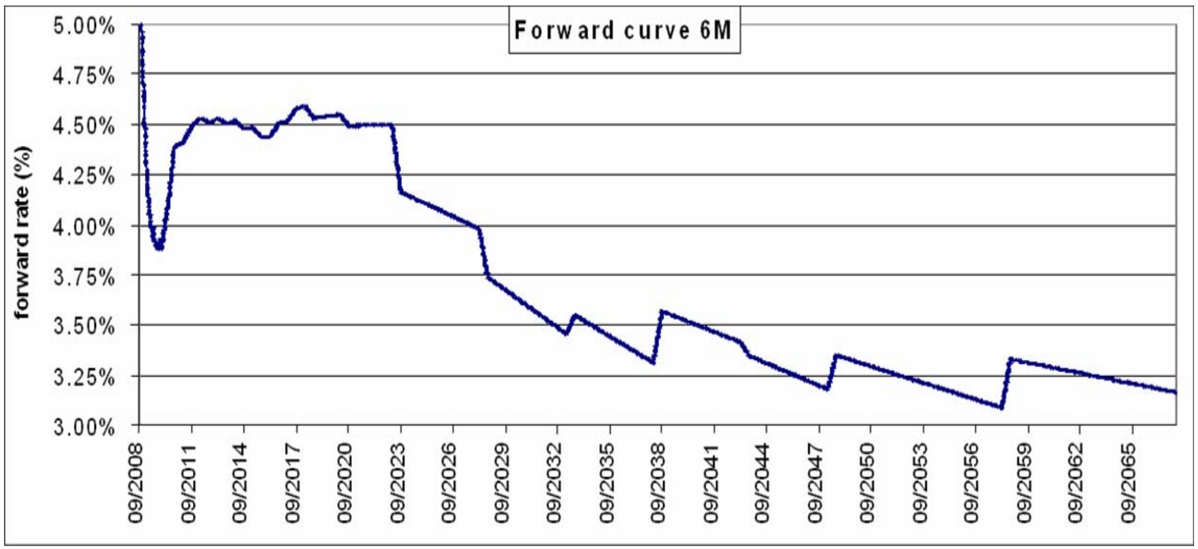






2: Double-Curve Framework:
Bad Curves





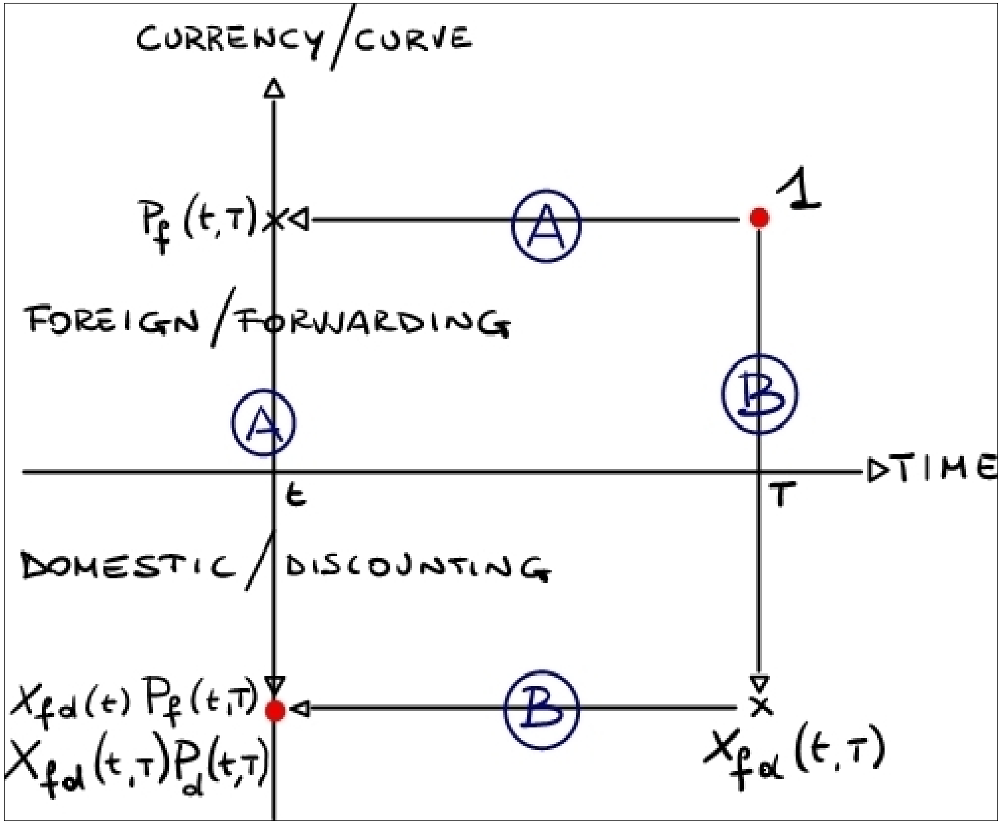
The same curves as before, but with linear interpolation.

The forward curve is discontinuous because linear interpolation is not derivable at market pillars.

The resulting basis adjustment curve is terrible.

Numerical results from QuantLib (*www.quantlib.org*)

3: Foreign Currency Analogy:
Forward vs Discount Basis Adjustment



1. Double-curve-double-currency:
   d = domestic, f = foreign

$$c_d(t) = x_{fd}(t)\, c_f(t),$$

$$X_{fd}(t,T)\, P_d(t,T) = x_{fd}(t)\, P_f(t,T),$$

2. Double-curve-single-currency:
   d = discounting, f = forwarding

$$x_{fd}(t) = 1,$$

$$X_{fd}(t,T) = \frac{P_f(t,T)}{P_d(t,T)},$$

this amounts to a **discount basis adjustment** for discount factors.

Forward vs discount basis adjustment:

$$BA_{fd}(t;T_1,T_2) = X_{fd}(t,T_2)\,\frac{\tau_f(T_1,T_2)}{\tau_d(T_1,T_2)}\,\frac{P_d(t,T_1) - P_d(t,T_2)}{P_d(t,T_1)\,X_{fd}(t,T_1) - P_d(t,T_2)\,X_{fd}(t,T_2)}.$$

3: Foreign Currency Analogy: *Quanto Adjustment*

1. Assume a lognormal martingale dynamic for the $\mathcal{C}_f$ **(foreign) forward rate**

$$\frac{dF_f(t;T_1,T_2)}{F_f(t;T_1,T_2)} = \sigma_f(t)\,dW_f^{T_2}(t), \quad Q_f^{T_2} \leftrightarrow P_f(t,T_2) \leftrightarrow C_f;$$

2. since $x_{fd}(t)P_f(t,T)$ is the price at time *t* of a $\mathcal{C}_d$ (domestic) tradable asset, the forward exchange rate must be a **martingale process**

$$\frac{dX_{fd}(t,T_2)}{X_{fd}(t,T_2)} = \sigma_X(t)\,dW_X^{T_2}(t), \quad Q_d^{T_2} \leftrightarrow P_d(t,T_2) \leftrightarrow C_d,$$

with
$$dW_f^{T_2}(t)\,dW_X^{T_2}(t) = \rho_{fX}(t)\,dt;$$

3. by changing numeraire from $\mathcal{C}_f$ to $\mathcal{C}_d$ we obtain the **modified dynamic**

$$\frac{dF_f(t;T_1,T_2)}{F_f(t;T_1,T_2)} = \mu_f(t)\,dt + \sigma_f(t)\,dW_f^{T_2}(t), \quad Q_d^{T_2} \leftrightarrow P_d(t,T_2) \leftrightarrow C_d,$$

$$\mu_f(t) = -\sigma_f(t)\sigma_X(t)\rho_{fX}(t);$$

4. and the modified expectation including the **quanto-adjustment**

$$E_t^{Q_d^{T_2}}\left[F_f(T_1;T_1,T_2)\right] = F_f(t;T_1,T_2)\,QA_{fd}\left(t;T_1,\sigma_f,\sigma_X,\rho_{fX}\right),$$

$$QA_{fd}\left(t;T_1,\sigma_f,\sigma_X,\rho_{fX}\right) = \exp\int_t^{T_1}\mu_f(s)\,ds = \exp\left[-\int_t^{T_1}\sigma_f(s)\sigma_X(s)\rho_{fX}(s)\,ds\right]$$

4: Pricing & Hedging IRD:
Pricing Plain Vanillas

1. FRA:

$$\mathbf{FRA}(t;T_1,T_2,K) = P_d(t,T_2)\tau_d(T_1,T_2)$$
$$\times\left[F_f(t;T_1,T_2)QA_{fd}\left(t,T_1,\sigma_f,\sigma_X,\rho_{fX}\right) - K\right],$$

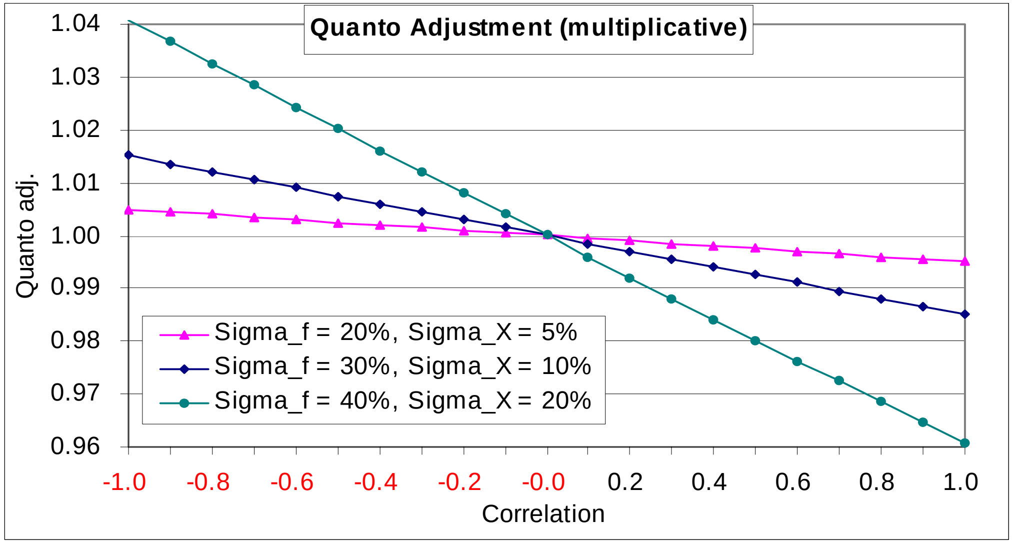
2. Swaps:

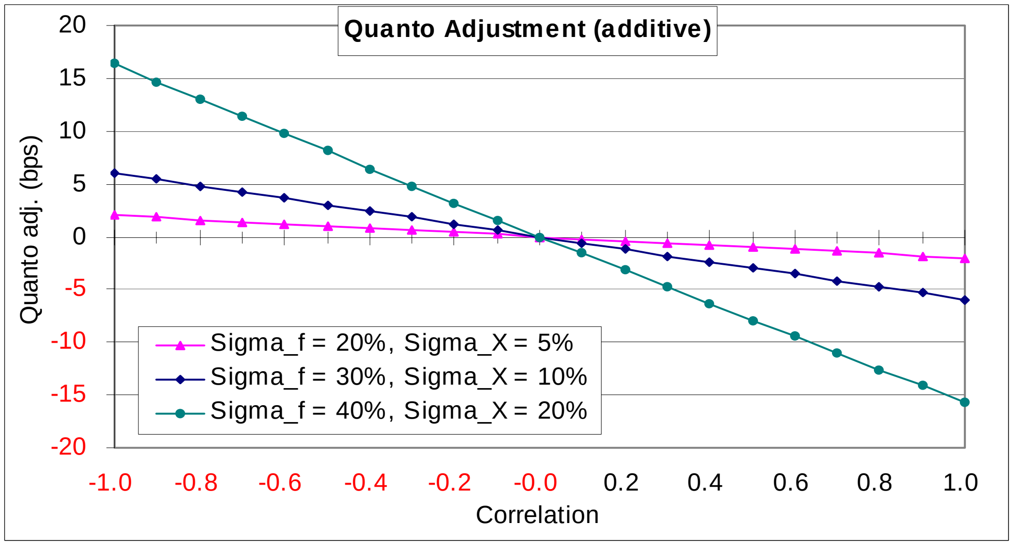
$$\mathbf{Swap}(t;\boldsymbol{T},\boldsymbol{K}) = \sum_{i=1}^{n} P_d(t,T_i)\tau_d(T_{i-1},T_i)$$
$$\times\left[F_f(t;T_{i-1},T_i)QA_{fd}\left(t,T_{i-1},\sigma_{f,i},\sigma_{X,i},\rho_{fX,i}\right) - K_i\right],$$

3. Caps/Floors:

$$\mathbf{CF}(t;\boldsymbol{T},\boldsymbol{K},\boldsymbol{\omega}) = \sum_{i=1}^{n} P_d(t,T_i)\tau_d(T_{i-1},T_i)$$
$$\times Bl\left[F_f(t;T_{i-1},T_i)QA_{fd}\left(t,T_{i-1},\sigma_{f,i},\sigma_{X,i},\rho_{fX,i}\right),K_i,\mu_{f,i},v_{f,i},\omega_i\right],$$
$$Bl[F,K,\mu,\sigma,\omega] = \omega\left[F\Phi\left(\omega d^+\right) - K\Phi\left(\omega d^-\right)\right],$$
$$d^{\pm} = \left[\ln\left(F/K\right) + \mu(t,T) \pm v^2(t,T)/2\right]/v(t,T),$$
$$\mu(t,T) = \int_t^T \mu(s)\,ds;\quad v^2(t,T) = \int_t^T \sigma^2(s)\,ds.$$

4: Pricing & Hedging IRD:
Pricing Plain Vanillas





For realistic values of volatilities and correlation **the adjustment may be not negligible** (time intervals longer than the 6M period used in figure further increase the effect). Notice that **positive correlation implies negative adjustment, thus lowering the forward rates** that enters the pricing formulas above. The standard market practice, with no quanto adjustment, is thus **not arbitrage free**. In practice the adjustment depends on market variables not directly quoted on the market, making virtually impossible to set up arbitrage positions and locking today positive gains in the future.

4: Pricing & Hedging IRD: *Hedging*

1. Given any portfolio of interest rate derivatives with price $\Pi\left(t,\boldsymbol{T},\boldsymbol{R}^{mkt}\right)$, compute delta risk with respect to **both curves** $\mathcal{C}_d$ **and** $\mathcal{C}_f$:

$$\Delta^{\pi}\left(t,\boldsymbol{T},\boldsymbol{R}^{mkt}\right)=\Delta_d^{\pi}\left(t,\boldsymbol{T},\boldsymbol{R}_d^{mkt}\right)+\Delta_f^{\pi}\left(t,\boldsymbol{T},\boldsymbol{R}_f^{mkt}\right)$$

$$=\sum_{j=1}^{N_d}\frac{\partial\Pi\left(t,\boldsymbol{T},\boldsymbol{R}^{mkt}\right)}{\partial R_{d,j}^{mkt}}+\sum_{j=1}^{N_f}\frac{\partial\Pi\left(t,\boldsymbol{T},\boldsymbol{R}^{mkt}\right)}{\partial R_{f,j}^{mkt}},$$

2. eventually aggregate it on the subset of most liquid market instruments (**hedging instruments**);

3. calculate **hedge ratios**:

$$h_{x,j}=\frac{\partial\Pi\left(t,\boldsymbol{T},\boldsymbol{R}^{mkt}\right)}{\partial R_{x,j}^{mkt}}\Bigg/\delta_{x,j}^{mkt},$$

$$\delta_{x,j}^{mkt}=\frac{\partial\pi_{x,j}^{mkt}(t)}{\partial R_{x,j}^{mkt}},\quad x=f,d.$$

5: Conclusions

1. We have reviewed the **pre and post credit crunch market practices for pricing & hedging interest rate derivatives**;
2. we have formalized the present **double-curve framework**, showing that **standard single-curve no arbitrage conditions are broken** and can be formally recovered with the introduction of a **basis adjustment**; numerical calculations reveals a **complex and oscillating micro-term structure**, testing the quality of the bootstrapping procedure (interpolation in particular), revealing the differences between different interest rate market areas, and influencing the price of similar interest rate instruments;
3. recurring to the foreign-currency analogy we have computed the **no arbitrage double-curve-single-currency pricing expressions** for basic interest rate derivatives, including a **quanto adjustment**, typical of cross currency derivatives, naturally arising from the change of numeraires naturally associated to the two yield curves. Numerical scenarios show that the **quanto adjustment can be non negligible, thus making the standard market practice, in principle, not arbitrage free**.

5: Future work

1. Numerical results for quanto adjustment with **more realistic volatilities and correlation** (implied and historical);

2. pricing **cross-currency swaps** using 5 curves;

5: Main references

[1] M. Bianchetti, “*Two Curves, One Price: Pricing & Hedging Interest Rate Derivatives Using Different Yield Curves for Discounting and Forwarding*”, (2009), available at SSRN: *http://ssrn.com/abstract=1334356*.

[2] M. Morini, "*Credit Modelling After the Subprime Crisis*", Marcus Evans course, 2008.

[3] F. Mercurio, "*Post Credit Crunch Interest Rates: Formulas and Market Models*", working paper, Bloomberg, 2008, available at SSRN: *http://ssrn.com/abstract=1332205.*

[4] D. Brigo, F. Mercurio, "*Interest Rate Models - Theory and Practice*", 2nd edition, Springer 2006.