Discrete Monitored Barrier Options: by Finite Difference Schemes
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2 Options with Discontinuous Payoff Function

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4 Difficulties in Pricing of Discrete Double Barrier Options

5 Crank-Nicolson Classic Scheme in Finance.

6 Works or Not for Options with Discontinuous Payoffs?

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Smooth and Discontinuous Payoff Function

✔ When the payoff function is a \textit{continuous function} there exist various ways determining the fair value for the option. For example, the Black-Scholes option valuation formula.

✔ However, in case of a option with a \textit{discontinuous payoff function}, the option valuation is more complicated.

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1. Class of Options with Discontinuous Payoff: Options with Discrete Monitoring

Figure: 1. Discrete double barrier knock-out call option. Payoff as initial function having discontinuities that is renewed at every monitoring date.
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Discrete Double Barrier Knock-out Payoff

Figure: 2. Payoff diagram of a discrete double barrier knock-out call option with discrete monitoring.
2. Class of Options with Discontinuous Payoff: Options with Jumps in the Payoff Function

Figure: 3. Cash-or-nothing call and put payoff diagrams.

Figure: 4. Supershare binary call payoff diagram.
Finite Difference Scheme Postulation

The option valuation problem reduces to solving the Black-Scholes equation for the underlying asset $V(S,t)$

$$\frac{\partial V}{\partial t} + rS \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0 \quad (1)$$

in the computational domain $[0,S_{max}] \times [0,T]$ and endowed

✔ with the boundary conditions: $V(0,t) = 0$, $V(S_{max},t) = 0$,

✔ barrier constraints

$$V(S,t) = \begin{cases} 
0 & \text{if } S \leq L \text{ or } S \geq U \text{ and } t \in B \\
V(S,t) & \text{otherwise}
\end{cases}$$

✔ and initial conditions (payoff for the original problem at $T$)

$$V(S,0) = \begin{cases} 
0 & \text{if } S \leq L \\
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Crank-Nicolson Classic Scheme Applied Discrete Double Barrier Knock-out Options

The finite difference approximation of the Black-Scholes equation provides the following linear system:

\[ PV_{n+1} = NV_n \quad (2) \]

with \( P \) and \( N \) the following tridiagonal matrices:

\[
P = \begin{cases} 
\frac{r}{4} \frac{S_j}{\Delta S} - \left( \frac{\sigma}{2} \frac{S_j}{\Delta S} \right)^2 & \frac{1}{\Delta t} + \frac{1}{2} \left( \frac{\sigma S_j}{\Delta S} \right)^2 + \frac{r}{2} ; - \frac{r}{4} \frac{S_j}{\Delta S} - \left( \frac{\sigma}{2} \frac{S_j}{\Delta S} \right)^2 \\
\end{cases}
\]

\[
N = \begin{cases} 
- \frac{r}{4} \frac{S_j}{\Delta S} + \left( \frac{\sigma}{2} \frac{S_j}{\Delta S} \right)^2 & \frac{1}{\Delta t} - \frac{1}{2} \left( \frac{\sigma S_j}{\Delta S} \right)^2 - \frac{r}{2} ; \frac{r}{4} \frac{S_j}{\Delta S} + \left( \frac{\sigma}{2} \frac{S_j}{\Delta S} \right)^2 \\
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Crank-Nicolson Method. Works or Not? Advantages and Disadvantages?

In general, the Crank-Nicolson method is a frequently used method in Finance because of the following two reasons:

✔ The method is second order-accurate in space discretization;
✔ Unconditionally stable in time, i.e. there is no time-step restriction;

Main drawbacks of the Crank-Nicolson method:

⋆ The positivity of the solution is not guaranteed;
⋆ The solution may suffer from undesired spurious oscillations;
⋆ Only the case $\sigma^2 > r$ could be analyzed but the case $\sigma^2 < r$ not.
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Crank-Nicolson Scheme Analysis when $\sigma^2 > r$.
Defining Eigenvalues of the Iteration Matrix

✔ The case $\sigma^2 < r$ could not be analyzed:

* Using the identity matrix $I$ $P$ and $N$ may be written as:

$$P = \frac{1}{\Delta t} I + C \quad \text{and} \quad N = \frac{1}{\Delta t} I - C,$$ where

$$C = \text{tridiag} \left\{ \frac{r}{4} \frac{S_j}{\Delta S} - \left( \frac{\sigma}{2} \frac{S_j}{\Delta S} \right)^2 ; \frac{1}{2} \left( \frac{\sigma S_j}{\Delta S} \right)^2 + \frac{r}{2} ; -\frac{r}{4} \frac{S_j}{\Delta S} - \left( \frac{\sigma}{2} \frac{S_j}{\Delta S} \right)^2 \right\}$$

* The tridiagonal matrix $C$ has positive diagonal elements and negative off-diagonal elements because we know that $\sigma^2 > r$.

Thus, the matrix $C$ is similar to a Jacobi matrix and admits distinct real eigenvalues $\lambda_i(C)$, see Ortega, [50].

According to the Gerschgorin theorem these $\lambda_i(C)$ are located in the interval $\left[ \frac{r}{2} ; \frac{r}{2} + (\sigma M)^2 \right]$, $M$ is the number of space steps.
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Factors For Oscillations Presence of Negative Eigenvalues of the Iteration Matrix Close to $-1$

✔ Diminishing the space step $\Delta S$, i.e. $M \to \infty$, then the spectrum $\rho(C) : = \max(\lambda_i(C)) \to \infty$ and $\lambda_i(P^{-1}N) = \frac{1-\Delta t \lambda_i(C)}{1+\Delta t \lambda_i(C)} \to -1$.

★ To the $M$ distinct eigenvalues $\lambda_i(P^{-1}N)$ could be associated $M$-number linearly independent eigenvectors $v_i$ that can be used as a basis for the $M$-dimensional space of the payoff $V(0)$ that is $V(0) = \sum_{i=1}^{M} d_i v_i$, where the $d_i$ are proper weights.

✔ As a consequence the numerical solution $V_{n+1}$ oscillates:

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★ Tavella describes this Crank-Nicolson phenomena as excitation of the eigenvalues of the finite difference matrix. He introduces the so called characteristic grid diffusion time $\tau_d : = \frac{\Delta S^2}{(\sigma S)^2}$ so that when $\Delta t \gg \tau_d$ oscillations may occur, [19].
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Factors For Oscillations Presence of Negative Eigenvalues of the Iteration Matrix Close to $-1$

- Diminishing the space step $\Delta S$, i.e. $M \to \infty$, then the spectrum $\rho(C) := \max(\lambda_i(C)) \to \infty$ and $\lambda_i(P^{-1}N) = \frac{1-\Delta t \lambda_i(C)}{1+\Delta t \lambda_i(C)} \to -1$.

- To the $M$ distinct eigenvalues $\lambda_i(P^{-1}N)$ could be associated $M$-number linearly independent eigenvectors $v_i$ that can be used as a basis for the $M$-dimensional space of the payoff $V(0)$ that is $V(0) = \sum_{i=1}^{M} d_i v_i$, where the $d_i$ are proper weights.

- As a consequence the numerical solution $V_{n+1}$ **oscillates**:

$$V_{n+1} = (P^{-1}N)^n V_0 = (P^{-1}N)^n \sum_{i=1}^{M} d_i v_i = \sum_{i=1}^{M} d_i (P^{-1}N)^n v_i = \sum_{i=1}^{M} d_i (\lambda_i)^n v_i$$

- Tavella describes this Crank-Nicolson phenomena as excitation of the eigenvalues of the finite difference matrix.

He introduces the so called **characteristic grid diffusion time** $\tau_d := \frac{\Delta S^2}{(\sigma S)^2}$ so that when $\Delta t \gg \tau_d$ oscillations may occur, [19].
Oscillations of the Crank-Nicolson Scheme. Discrete Double Barrier Knock-out Option

Figure: 5. Option pricing just before the last monitoring date $t_F = 12$. $L = 90$, $K = 100$, $U = 110$, $r = 0.05$, $\sigma = 0.2$, $T = 1$, $\Delta S = 0.1$, $\Delta t = 0.01$. Financial provisions: 1. Discontinuous Payoff; 2. Parameters $\sigma^2 < r$.
Sufficient Conditions For Lack of Oscillations. Iteration Matrix With Positive Eigenvalues

★ The case of low volatility, i.e. $\sigma^2 < r$, could not be examined.

✔ Thus, if $\sigma^2 > r$, the spurious oscillation may disappear if the spectrum of the iteration matrix $P^{-1}N$ contains only positive eigenvalues, or negative eigenvalues far from -1.

✔ The last condition $\Delta t < \frac{2}{r+2(\sigma M)^2}$ is sufficient condition for avoiding oscillations but it makes the Crank-Nicolson scheme conditionally stable. If this time-step restriction is not preserved, the numerical solution may oscillate.
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A sufficient condition for stable and free of oscillations solution is

$$\lambda_i(P^{-1}N) = \frac{1 - \Delta t \lambda_i(C)}{1 + \Delta t \lambda_i(C)} \in (0, 1) \iff 1 - \Delta t \lambda_i(C) > 0$$

leading to

$$\Delta t < \frac{1}{\lambda_i(C)} < \frac{1}{\rho(C)} < \frac{2}{r + 2(\sigma M)^2}$$

where $\rho(C) := \max(\lambda_i(C))$.

The last condition $\Delta t < \frac{2}{r + 2(\sigma M)^2}$ is sufficient condition for avoiding oscillations but it makes the Crank-Nicolson scheme conditionally stable. If this time-step restriction is not preserved, the numerical solution may oscillate.
**Sufficient Conditions For Lack of Oscillations.**

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✔ The last condition $\Delta t < \frac{2}{r + 2(\sigma M)^2}$ is **sufficient condition for avoiding oscillations** but it makes the Crank-Nicolson scheme **conditionally stable**. If this **time-step restriction** is not preserved, the numerical solution may **oscillate**.
The case of low volatility, i.e. $\sigma^2 < r$, could not be examined.

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The Crank-Nicolson Iteration Matrix $P^{-1}N$. Sufficient Conditions for Positive Solution

* For fixed $\Delta t$, when $M \to \infty$, i.e. $\Delta S \to \infty$, $N$ could be negative:

$$N = \text{tridiag}\left\{ -\frac{r}{4} j + \left(\frac{\sigma}{2} j\right)^2 ; \frac{1}{\Delta t} - \frac{1}{2} (\sigma j)^2 - \frac{r}{2} ; \frac{r}{4} j + \left(\frac{\sigma}{2} j\right)^2 \right\}$$

* Thus, the positivity of the solution $V_{n+1}$ is not guaranteed because it is possible to happen that $V_{n+1} = (P^{-1}N)V_n < 0$. 

Sufficient conditions for positive solution. Positive matrix $P^{-1}N$. 

- The hypothesis $\sigma^2 > r$ is assumed;
- Let $\Delta t < \frac{2r + (\sigma M)^2}{r^2}$.

Then the matrix $P$ is an irreducible diagonally dominant matrix. Thus $P$ is an M-matrix and $P^{-1} \succ 0$. Moreover, $|P^{-1}|_{\infty} \leq \frac{1}{1 - \Delta t + \frac{r^2}{2}}$.

Both positivity and discrete maximum principle are satisfied.

$\star$ From $\Delta t < \frac{2r + (\sigma M)^2}{r^2} \Rightarrow N > 0$. And from $P^{-1} \succ 0 \Rightarrow A = P^{-1}N > 0$.

Then $V_n = AV_n - 1 = A(AV_n - 2) = \ldots = A^n V_0$ is positive since $V_0 \geq 0$. 

$\star$ $|V_{n+1}|_{\infty} = |(P^{-1}N)V_n|_{\infty} = |P^{-1}|_{\infty}|N|_{\infty}|V_n|_{\infty} \leq \frac{1}{1 - \Delta t + \frac{r^2}{2}}|V_n|_{\infty}$.
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1. The hypothesis $\sigma^2 > r$ is assumed;
2. Let $\Delta t < \frac{2}{r + (\sigma M)^2}$. 

From $\Delta t < \frac{2}{r + (\sigma M)^2}$, then $N > 0$. And from $P^{-1}N > 0 \Rightarrow A = P^{-1}N > 0$.

Then $V_{n+1} = AV_n - 1 = A(A V_n - 2) = \ldots = A^n V_0$ is positive since $V_0 \geq 0$. 

$$||V_{n+1}||_\infty = ||(P^{-1}N) V_n||_\infty \leq ||P^{-1}||_\infty |N|_\infty |V_n||_\infty \leq 1 \frac{\Delta t}{r} + \frac{r^2}{1 \Delta t}$$

$$||V_n||_\infty \leq \frac{1}{1 - \frac{\Delta t}{r}}$$
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Options with discontinuous payoffs makes the direct application of the Crank-Nicolson method inefficient because:

- The hypothesis $\sigma^2 > r$ for the parameters $\sigma$ and $r$ is assumed;
- Conditional stability $\Delta t < \frac{1}{2(r + \sigma^2)}$, i.e., a time-step restriction, implying uniquely positive eigenvalues of the iteration matrix.

Thus the classic Crank-Nicolson scheme works successfully and the numerical solution is positive and identifies as a maximum principle.
Crank-Nicolson Scheme

★ Options with **discontinuous payoffs** makes the *direct application* of the Crank-Nicolson method *inefficient* because:

**Sufficient conditions for working successfully the scheme**

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Modified Crank-Nicolson Scheme. A Variant for Discretization of the Reaction Term $-rV$

Figure: 6. Involved nodes in the Crank-Nicolson variant scheme.

★ It is shown that the reaction term $-rV$ is discretized by six adjacent nodes through the standard procedure for $V(S,t)$

$$a(V_{j-1}^n + V_{j+1}^n) + b(V_{j-1}^{n+1} + V_{j+1}^{n+1}) + \left(\frac{1}{2} - a - b\right)(V_{j}^{n+1} + V_{j}^{n})$$

with a discretization error $o((\Delta S)^2, (\Delta t)^2)$ if $a = b$, i.e. the second order accuracy is preserved, and $o(\Delta S, (\Delta t)^2)$ if $a \neq b$.

★ Here $a$ and $b$ are arbitrary constants to be determined below.

★ If $a = b = 0$, it leads to the classic Crank-Nicolson scheme.
Modified Crank-Nicolson Scheme. A Variant for Discretization of the Reaction Term $-rV$

![Figure: 6. Involved nodes in the Crank-Nicolson variant scheme.](image)

- It is shown that the reaction term $-rV$ is discretized by six adjacent nodes through the standard procedure for $V(S,t)$

\[
a\left(V_{j-1}^n + V_{j+1}^n\right) + b\left(V_{j-1}^{n+1} + V_{j+1}^{n+1}\right) + \left(\frac{1}{2} - a - b\right)\left(V_{j}^{n+1} + V_{j}^{n}\right)
\]

with a discretization error $o((\Delta S)^2, (\Delta t)^2)$ if $a = b$, i.e. the second order accuracy is preserved, and $o(\Delta S, (\Delta t)^2)$ if $a \neq b$.

- Here $a$ and $b$ are arbitrary constants to be determined below.

- If $a = b = 0$, it leads to the classic Crank-Nicolson scheme.
Modified Crank-Nicolson Scheme. A Variant for Discretization of the Reaction Term $-rV$

Figure: 6. Involved nodes in the Crank-Nicolson variant scheme.

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$$ a(V_{j-1}^n + V_{j+1}^n) + b(V_{j-1}^{n+1} + V_{j+1}^{n+1}) + \left(\frac{1}{2} - a - b\right)(V_j^{n+1} + V_j^n) $$

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Discrete Monitored Barrier Options: by Finite Difference Schemes

Aldo Tagliani, Faculty of Economics
Ph D Mariyan Milev, Faculty of Mathematics

The Main Problem
Options with Discontinuous Payoff Function
Option Valuation Problem. Postulation.
Difficulties in Pricing of Discrete Double Barrier Options
Crank-Nicolson Classic Scheme in Finance.

Works or Not for Options with Discontinuous Payoffs?

Crank-Nicolson Classic and Variant Schemes. Reaction Term Discretization. Comparison

Figure: 7. The upper represent involved nodes in the New Variant of Crank-Nicolson scheme. The lower figure of the frequently used standard Crank-Nicolson schemes in Finance.
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Crank-Nicolson Variant Scheme. Analysis

★ The finite difference approximation of the Black-Scholes equation provides the linear system:

\[ PV_{n+1} = NV_n \]  \hspace{1cm} (3)

with \( P \) and \( N \) the following tridiagonal matrices:

\[
P = \left\{ rb + \frac{r}{4} j - \frac{(\sigma j)^2}{4}; \frac{1}{\Delta t} + \frac{(\sigma j)^2}{2} + r \left( \frac{1}{2} - a - b \right); rb - \frac{r}{4} j - \frac{(\sigma j)^2}{4} \right\}
\]

\[
N = \left\{ \frac{(\sigma j)^2}{4} - ra - \frac{r}{4} j; \frac{1}{\Delta t} - \frac{(\sigma j)^2}{2} - r \left( \frac{1}{2} - a - b \right); \frac{(\sigma j)^2}{4} - ra + \frac{r}{4} j \right\}
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★ If \( a = b = 0 \) is chosen then the standard second-order accurate Crank-Nicolson scheme is obtained.

✔ But how will be defined the arbitrary parameters \( a \) and \( b \)?
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Crank-Nicolson Variant Scheme Parameters. Sufficient Conditions for Working Successfully

- In order to be satisfied all the financial requirements of the contract, the new variant of the Crank-Nicolson numerical scheme has been investigated where:

  ✔ The numerical solution $V(S,t)$ is positive;
  ✔ The discrete version of the maximum principle is satisfied;
  ✔ The spectrum of the iteration matrix $P^{-1}N$ in the equation $V_{n+1} = P^{-1}NV_n$ contains uniquely positive eigenvalues.
  ✔ The Crank-Nicolson variant scheme is independent of the parameters condition $\sigma^2 < r$ or $\sigma^2 > r$.
  ✔ All this criteria could be satisfied using the fact that in the discretization of the Black-Scholes equation by the modified Crank-Nicolson scheme there are two free parameters $a, b$ that could be defined, i.e. there are two degrees of freedom.
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Crank-Nicolson Variant Scheme. M-matrices. Positive Solution. Defining Parameters $a$ and $b$

The solution verifies two criteria if M-matrices are used:

- ✔ The iteration matrix $P^{-1}N$ of the scheme is positive and thus the positive numerical solution $V(S,t)$ could be guaranteed;
- ✔ The solution verifies the discrete maximum principle;

Sufficient conditions the iteration matrix $P^{-1}N$ to be positive:

- ✔ $P^{-1} > 0$, that is true if $P$ is an M-matrix leading to: $b \leq -\frac{r}{16\sigma^2}$;
- ✔ $N > 0$, that leads to the condition: $a \leq -\frac{r}{16\sigma^2}$;

Thus, having in mind the discretization error $o((\Delta S)^2, (\Delta t)^2)$ of this scheme if $a = b$, in order to obtain a second order accurate scheme we define the two parameters $a$ and $b$ as:

$$a = b = -\frac{r}{16 \sigma^2},$$

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★ Analogously the Crank-Nicolson scheme analysis, using
\[ P = \frac{1}{\Delta t} I + C, \quad N = \frac{1}{\Delta t} I - C, \quad \lambda_i(P^{-1}N) = \frac{1-\Delta t \lambda_i(C)}{1+\Delta t \lambda_i(C)}, \] we prove that:

The iteration matrix \( P^{-1}N \) of the Crank-Nicolson variant scheme admits \( M \) real distinct and positive eigenvalues \( \lambda_i(P^{-1}N) \) and \( \lambda_i(P^{-1}N) \in (0, 1), i = 1, 2, \ldots, M \) if \( \Delta t < \frac{1}{r(\frac{1}{2} - 4b) + (\sigma M)^2}. \)

✔ Then the absence of oscillations requires a time-step restriction.

✔ However, the advantage of the Crank-Nicolson variant scheme is that it works successfully both for the cases \( \sigma^2 > r \) and \( \sigma^2 < r \). The variant scheme is conditionally stable because \( \Delta t < \Delta t_1: \)

\[ \Delta t < \Delta t_1 = \frac{1}{r(\frac{1}{2} - 4b) + (\sigma M)^2} \]

★ For large \( M \) values, we notice that \( \Delta t_1 \approx \frac{1}{(\sigma M)^2} =: \tau_d \), i.e. the characteristic grid diffusion time introduced by Tavella, [19].

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Then the absence of oscillations requires a *time-step restriction*.

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Crank-Nicolson Variant Scheme. Damping Discontinuous Initial Condition. $V_n = S\Lambda^n S^{-1}V_0$

Figure: 9. Crank-Nicolson variant scheme damp the high-frequency terms of the discontinuous initial function in case of a discrete double barrier knock-out call option and there are not spurious oscillations.
Conditional Stability of the Crank-Nicolson Variant Scheme. Discrete Maximum Principle

\[ \frac{1}{\Delta t} - \frac{(\sigma j)^2}{2} - r\left(\frac{1}{2} - a - b\right) > 0 \iff \Delta t < \frac{1}{r\left(\frac{1}{2} - a - b\right) + \frac{1}{2}(\sigma M)^2} \]

\[ \star \text{ Remark: All diagonal elements of the matrix } N \text{ are positive if} \]

\[ \frac{1}{\Delta t} - \frac{(\sigma j)^2}{2} - r\left(\frac{1}{2} - a - b\right) > 0 \iff \Delta t < \frac{1}{r\left(\frac{1}{2} - a - b\right) + \frac{1}{2}(\sigma M)^2} \]

\[ \star \text{ Thus, positivity of the matrix } N \text{ is another factor except the uniquely positive eigenvalues of the iteration matrix that requires conditional stability of the Crank-Nicolson variant scheme.} \]

\[ \star \text{ The matrix } P \text{ is strongly row diagonally dominant and it is true that } \|P^{-1}\|_{\infty} \leq \left(\frac{1}{\Delta t} + \frac{r}{2}\right)^{-1}, \text{ Windish, [87].} \]

\[ \checkmark \text{ Using the norms } \|P^{-1}\|_{\infty} \leq \left(\frac{1}{\Delta t} + \frac{r}{2}\right)^{-1} \text{ and } \|N\|_{\infty} = \frac{1}{\Delta t} - \frac{r}{2}, \]

we verify that the solution of the variant scheme satisfies the discrete maximum principle: \[ \|V_{n+1}\|_{\infty} = \|(P^{-1}N)V_{n}\|_{\infty} \]

\[ \|V_{n+1}\|_{\infty} = \|P^{-1}\|_{\infty}\|N\|_{\infty}\|V_{n}\|_{\infty} \leq \frac{1}{\Delta t} - \frac{r}{2} \|V_{n}\|_{\infty} \leq \|V_{n}\|_{\infty} \]

\[ \|V_{n+1}\|_{\infty} = \|P^{-1}\|_{\infty}\|N\|_{\infty}\|V_{n}\|_{\infty} \leq \frac{1}{\Delta t} - \frac{r}{2} \|V_{n}\|_{\infty} \leq \|V_{n}\|_{\infty} \]
**Remark:** All diagonal elements of the matrix $N$ are positive if

$$\frac{1}{\Delta t} - \frac{(\sigma j)^2}{2} - r\left(\frac{1}{2} - a - b\right) > 0 \iff \Delta t < \frac{1}{r\left(\frac{1}{2} - a - b\right) + \frac{1}{2}(\sigma M)^2}$$

Thus, positivity of the matrix $N$ is another factor except the *uniquely positive eigenvalues of the iteration matrix* that requires conditional stability of the Crank-Nicolson variant scheme.

The matrix $P$ is strongly row diagonally dominant and it is true that $||P^{-1}||_\infty \leq \left(\frac{1}{\Delta t} + \frac{r}{2}\right)^{-1}$, Windish, [87].

Using the norms $||P^{-1}||_\infty \leq \left(\frac{1}{\Delta t} + \frac{r}{2}\right)^{-1}$ and $||N||_\infty = \frac{1}{\Delta t} - \frac{r}{2}$, we verify that the solution of the variant scheme satisfies the discrete maximum principle: $||V_{n+1}||_\infty = ||(P^{-1}N)V_n||_\infty$

$$||V_{n+1}||_\infty = ||P^{-1}||_\infty||N||_\infty||V_n||_\infty \leq \frac{1}{\Delta t} - \frac{r}{2}||V_n||_\infty \leq ||V_n||_\infty$$
Conditional Stability of the Crank-Nicolson Variant Scheme. Discrete Maximum Principle

★ **Remark:** All diagonal elements of the matrix $N$ are positive if

$$\frac{1}{\Delta t} - \frac{(\sigma j)^2}{2} - r\left(\frac{1}{2} - a - b\right) > 0 \iff \Delta t < \frac{1}{r\left(\frac{1}{2} - a - b\right) + \frac{1}{2}(\sigma M)^2}$$

★ Thus, positivity of the matrix $N$ is another factor except the *uniquely positive eigenvalues of the iteration matrix* that requires *conditional stability* of the Crank-Nicolson variant scheme.

★ The matrix $P$ is strongly row diagonally dominant and it is true that $||P^{-1}||_\infty \leq \left(\frac{1}{\Delta t} + \frac{r}{2}\right)^{-1}$, Windish, [87].

✔ Using the norms $||P^{-1}||_\infty \leq \left(\frac{1}{\Delta t} + \frac{r}{2}\right)^{-1}$ and $||N||_\infty = \frac{1}{\Delta t} - \frac{r}{2}$, we verify that the solution of the variant scheme satisfies the discrete maximum principle: $||V_{n+1}||_\infty = ||(P^{-1}N)V_n||_\infty$

$$||V_{n+1}||_\infty = ||P^{-1}||_\infty ||N||_\infty ||V_n||_\infty \leq \frac{1}{\Delta t} - \frac{r}{2} ||V_n||_\infty \leq ||V_n||_\infty$$
Conditional Stability of the Crank-Nicolson Variant Scheme. Discrete Maximum Principle

* Remark: All diagonal elements of the matrix $N$ are positive if

$$\frac{1}{\Delta t} - \frac{(\sigma j)^2}{2} - r\left(\frac{1}{2} - a - b\right) > 0 \iff \Delta t < \frac{1}{r\left(\frac{1}{2} - a - b\right) + \frac{1}{2} (\sigma M)^2}$$

* Thus, positivity of the matrix $N$ is another factor except the uniquely positive eigenvalues of the iteration matrix that requires conditional stability of the Crank-Nicolson variant scheme.

* The matrix $P$ is strongly row diagonally dominant and it is true that $\|P^{-1}\|_{\infty} \leq \left(\frac{1}{\Delta t} + \frac{r}{2}\right)^{-1}$, Windish, [87].

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$$\|V_{n+1}\|_{\infty} = \|P^{-1}\|_{\infty}\|N\|_{\infty}\|V_n\|_{\infty} \leq \frac{1}{\Delta t} - \frac{r}{2} \|V_n\|_{\infty} \leq \|V_n\|_{\infty}$$
Discrete Monitored Barrier Options: by Finite Difference Schemes

Aldo Tagliani, Faculty of Economics
Ph D Mariyan Milev, Faculty of Mathematics

The Main Problem
Options with Discontinuous Payoff Function

Difficulties in Pricing of Discrete Double Barrier Options
Crank-Nicolson Classic Scheme in Finance.

Works or Not for Options with Discontinuous Payoffs?

Figure: 10. \( A = 1, K = 10, r = 0.05, \sigma = 0.01, T = 1, \Delta S = 0.005, \Delta t = 0.0001 \)
Modified Semi-implicit Scheme. Supershare Binary Call Option and Its Delta and Gamma

Figure: 11. $K = 10, d = 5, r = 0.05, \sigma = 0.01, T = 1; \Delta S = 0.005, \Delta t = 0.0001$
We have proposed a variant of the Crank-Nicolson scheme that:

✔ guarantees positive solution;
✔ it is free of spurious oscillations but it is conditionally stable;
✔ and it is independent of the financial parameters, i.e. $\sigma^2$ and $r$;
Conclusions

We have proposed a variant of the Crank-Nicolson scheme that:

✔ guarantees positive solution;
✔ it is free of spurious oscillations but it is conditionally stable
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Thank you!

Discrete Monitored Barrier Options: by Finite Difference Schemes

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The Main Problem

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Option Valuation Problem
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Crank-Nicolson Classic Scheme in Finance.
Works or Not for Options with Discontinuous Payoffs?

Finite Difference Schemes. Comparison

<table>
<thead>
<tr>
<th>Stock Price (S)</th>
<th>Option Value (V)</th>
</tr>
</thead>
<tbody>
<tr>
<td>95</td>
<td>0.15</td>
</tr>
<tr>
<td>100</td>
<td>0.20</td>
</tr>
<tr>
<td>105</td>
<td>0.23</td>
</tr>
<tr>
<td>110</td>
<td>0.24</td>
</tr>
</tbody>
</table>

Duffy exponentially fitted scheme
Semi-implicit upwind scheme
Crank–Nicolson variant scheme
Numerical algorithm 1000 points
Finite Difference Results

<table>
<thead>
<tr>
<th>Asset Price $S_0$</th>
<th>Crank Nicolson Scheme</th>
<th>Standard Implicit Scheme</th>
<th>Semi-impl. Upwind Scheme</th>
<th>Implicit Upwind Scheme</th>
<th>Monte Carlo method (st. error) $10^7$-asset paths</th>
</tr>
</thead>
<tbody>
<tr>
<td>95.0</td>
<td>0.165630</td>
<td>0.165639</td>
<td>0.173203</td>
<td>0.173055</td>
<td>–</td>
</tr>
<tr>
<td>95.5</td>
<td>0.173208</td>
<td>0.173218</td>
<td>0.181141</td>
<td>0.179820</td>
<td>0.18291 (0.00066)</td>
</tr>
<tr>
<td>99.5</td>
<td>0.218163</td>
<td>0.218180</td>
<td>0.228179</td>
<td>0.227519</td>
<td>0.22923 (0.00073)</td>
</tr>
<tr>
<td>100.0</td>
<td>0.221212</td>
<td>0.221228</td>
<td>0.231348</td>
<td>0.231232</td>
<td>0.23263 (0.00036)</td>
</tr>
<tr>
<td>100.5</td>
<td>0.223568</td>
<td>0.223586</td>
<td>0.233819</td>
<td>0.233384</td>
<td>0.23410 (0.00073)</td>
</tr>
<tr>
<td>109.5</td>
<td>0.165807</td>
<td>0.165817</td>
<td>0.173108</td>
<td>0.174193</td>
<td>0.17426 (0.00063)</td>
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<tr>
<td>110.0</td>
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<td>0.159065</td>
<td>0.166037</td>
<td>0.166075</td>
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<td>594 sec.</td>
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<td>119 sec.</td>
<td>347 sec.</td>
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</table>

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<tr>
<td>95.0</td>
<td>0.017427</td>
<td>0.017432</td>
<td>0.019516</td>
<td>0.019481</td>
<td>–</td>
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<tr>
<td>95.5</td>
<td>0.019918</td>
<td>0.019924</td>
<td>0.022308</td>
<td>0.021847</td>
<td>0.022305 (0.00049)</td>
</tr>
<tr>
<td>99.5</td>
<td>0.037595</td>
<td>0.037608</td>
<td>0.041881</td>
<td>0.041594</td>
<td>0.041465 (0.00062)</td>
</tr>
<tr>
<td>100.0</td>
<td>0.038950</td>
<td>0.038963</td>
<td>0.043348</td>
<td>0.043310</td>
<td>0.042736(0.00031)</td>
</tr>
<tr>
<td>100.5</td>
<td>0.040021</td>
<td>0.040035</td>
<td>0.044514</td>
<td>0.044322</td>
<td>0.043934 (0.00061)</td>
</tr>
<tr>
<td>109.5</td>
<td>0.018860</td>
<td>0.018865</td>
<td>0.021019</td>
<td>0.021394</td>
<td>0.020938 (0.00043)</td>
</tr>
<tr>
<td>110.0</td>
<td>0.016741</td>
<td>0.016745</td>
<td>0.018650</td>
<td>0.018662</td>
<td>–</td>
</tr>
<tr>
<td>CPU</td>
<td>900 sec.</td>
<td>615 sec.</td>
<td>162 sec.</td>
<td>123 sec.</td>
<td>403 sec.</td>
</tr>
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Numerical Results. High Barrier Observation Frequency: Daily and Weekly Monitoring

<table>
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<tr>
<td>3.00312 172 sec.</td>
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</table>

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<th>Table 9: Barrier observation frequency (125 times) - daily</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.492991 191 sec.</td>
</tr>
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</table>

Value of the underlying asset $S_0 = 100$, $K = 100$, $\sigma = 0.2$, $T = 0.5$, $r = 0.1$, $L = 95$, $U = 125$.

The parameters of the semi-implicit upwind scheme are $\Delta S = 0.05$, $\Delta t = 0.001$, and for the implicit-upwind scheme are $\Delta S = 0.05$, $\Delta t = 0.001$, the cautelative value is $S_{max} = 200$.

★ It is difficult to be defined whether the semi-implicit or implicit-upwind scheme is more accurate. For example:

★ The semi-implicit option value 3.00312 in case of *weekly monitoring* is much closer to the results obtained by other numerical methods than the implicit upwind value.

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**Figure: 12.** The Black-Scholes surface of a discrete double barrier knock-out call option monitored monthly (6 times) with parameters $K = 100, \sigma = 0.2, T = 0.5, r = 0.1, L = 95, U = 140$. Prices of discrete double knock-out call option for value of the underlying asset $S_0 = 100$. 
Future Research

✔ Valuation of Asian and Lookback Options
✔ Valuation of American Options
✔ Acceleration Methods. Time-stepping
✔ Probabilistic Valuation of Options
✔ Valuation of Bermudian and Shout Options
Future Research

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Central and Upwind Schemes

★ In the first two finite difference schemes the first derivative, i.e the Greek Delta, is approximated by a central difference while in the third and the forth by a upwind difference.

★ All the schemes satisfy the discrete maximum principle, have spectrum of the iteration matrix that contains uniquely positive eigenvalues, and the numerical solution is positive.

★ Thus, the schemes have different restriction of the time step for evitating spurious oscillations of the numerical solution.

★ The choice of the scheme depends on several factors:

✔ the relation of the interest rate and volatility parameters, i.e. whether $\sigma^2 > r$ or $\sigma^2 < r$ is true. The second case remains still unsolved when applied the Crank-Nicolson method.

✔ the distance of each of the barriers to the strike price.

✔ whether barriers assume high values or not.

✔ monitoring frequency, see the next example.
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Barriers Assume High Values. Results

<table>
<thead>
<tr>
<th>Semi-impl. Upwind Scheme</th>
<th>Implicit Upwind Scheme</th>
<th>Zvan Implicit Scheme</th>
<th>Cheuk-Vorst Trinomial Tree</th>
<th>HOBIS Implicit Scheme</th>
<th>M.Carlo $10^7$-paths (st.error)</th>
</tr>
</thead>
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<tr>
<td>3.00312</td>
<td>2.998638</td>
<td>3.012</td>
<td>2.989</td>
<td>3.006 hundreds.</td>
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<tr>
<td>172 sec.</td>
<td>123 sec.</td>
<td>9.47 sec.</td>
<td>–</td>
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| Table 8: Barrier observation frequency (25 times) - weekly |

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<tr>
<td>2.492991</td>
<td>2.488748</td>
<td>2.485</td>
<td>2.482</td>
<td>2.482 hundreds</td>
<td>2.48142 (0.0032)</td>
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<tr>
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<td>134 sec.</td>
<td>37.93 sec.</td>
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| Table 9: Barrier observation frequency (125 times) - daily |

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<table>
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<tr>
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<td>9.47 sec.</td>
<td>–</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 9: Barrier observation frequency (125 times) - daily

<table>
<thead>
<tr>
<th>Scheme Type</th>
<th>Semi-impl. Upwind Scheme</th>
<th>Implicit Scheme</th>
<th>Zvan Implicit Scheme</th>
<th>Cheuk-Vorst Trinomial Tree</th>
<th>HOBIS Implicit Scheme</th>
<th>M.Carlo 10^7-paths (st.error)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Semi-impl. Upwind Scheme</td>
<td>2.492991</td>
<td>2.488748</td>
<td>2.485</td>
<td>2.482</td>
<td>2.482 hundreds</td>
<td>2.48142 (0.0032)</td>
</tr>
<tr>
<td></td>
<td>191 sec.</td>
<td>134 sec.</td>
<td>37.93 sec.</td>
<td>–</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Value of the underlying asset \( S_0 = 100, K = 100, \sigma = 0.2, T = 0.5, r = 0.1, L = 95, U = 125 \).
Barriers Assume High Values. Results

<table>
<thead>
<tr>
<th>Table 8: Barrier observation frequency (25 times) - weekly</th>
</tr>
</thead>
<tbody>
<tr>
<td>Semi-impl. Upwind Scheme</td>
</tr>
<tr>
<td>---------------------------</td>
</tr>
<tr>
<td>3.00312</td>
</tr>
<tr>
<td>172 sec.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 9: Barrier observation frequency (125 times) - daily</th>
</tr>
</thead>
<tbody>
<tr>
<td>Semi-impl. Upwind Scheme</td>
</tr>
<tr>
<td>---------------------------</td>
</tr>
<tr>
<td>2.492991</td>
</tr>
<tr>
<td>191 sec.</td>
</tr>
</tbody>
</table>

Value of the underlying asset $S_0 = 100$, $K = 100$, $\sigma = 0.2$, $T = 0.5$, $r = 0.1$, $L = 95$, $U = 125$.

The parameters of the semi-implicit upwind scheme are $\Delta S = 0.05$, $\Delta t = 0.001$, and for the implicit-upwind scheme are $\Delta S = 0.05$, $\Delta t = 0.001$, the cautelative value is $S_{max} = 200$.

★ It is difficult to be defined whether the semi-implicit or implicit-upwind scheme is more accurate. For example:
★ The semi-implicit option value 3.00312 in case of weekly monitoring is much closer to the results obtained by other numerical methods than the implicit upwind value.
★ However, the implicit-upwind option value 2.488748 is more precise in case of daily monitoring.
Observations and Conclusions

★ The Crank-Nicolson variant scheme differs from the standard Crank-Nicolson scheme because its application does not depend of the parameter condition $\sigma^2 > r$.

★ For discrete double barrier options the semi-implicit variant scheme is much more accurate and faster, Tables 4,5.

★ Semi-implicit and implicit-upwind difference scheme give better results than the Crank-Nicolson scheme in case the barriers assume high values - the case more difficult for discretization.

★ The numerical solution of upwind schemes does not suffer of spurious oscillation in the region close to the barriers.

★ The time step restriction for stability of upwind schemes is not so strong as the Crank-Nicolson one and permits using more space points, i.e 4000 and $\Delta S = 0.05, \Delta t = 0.001$.

★ In this case implicit-upwind finite difference schemes are much more faster than central implicit schemes.
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Second Example and Conclusions

★ In the second experiment we compare the finite difference results when the two barriers *assume small values*, i.e. $L = 4$ and $U = 12$. Parameters: $K = 8$, $\sigma = 0.25$, $r = 0.05$, $T = 0.5$. We could make the following conclusions:

★ The first order implicit variant scheme and the Duffy exponentially fitted scheme are much more accurate and faster than the semi-implicit central scheme.

★ The Crank-Nicolson variant scheme is second-order accurate both in time and space and naturally gives more accurate results than most standard first-order accurate schemes.

★ It should be remembered that in case of pricing options with *discontinuous payoffs*, the standard second-order accurate Crank-Nicolson scheme often suffers from undesired *spurious oscillations* or gives inaccurate numerical results, see Table 7.
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## Analysis of Finite Difference Scheme Accuracy

### Table 10: Prices of discrete double barrier knock-out call option monitored 5 times.

<table>
<thead>
<tr>
<th>Underl. Asset $S_0$</th>
<th>Implicit Variant Scheme</th>
<th>Duffy Implicit Scheme</th>
<th>Crank-Nicolson Variant</th>
<th>Semi-impl. Central Scheme</th>
<th>Monte Carlo Simulation (st. error) 10^7-asset paths</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>1.92.10^{-5}</td>
<td>1.91.10^{-5}</td>
<td>1.80.10^{-5}</td>
<td>3.34.10^{-6}</td>
<td>–</td>
</tr>
<tr>
<td>4.005</td>
<td>1.97.10^{-5}</td>
<td>1.97.10^{-5}</td>
<td>1.92.10^{-5}</td>
<td>3.46.10^{-6}</td>
<td>1.98.10^{-5} (0.00004)</td>
</tr>
<tr>
<td>8</td>
<td>0.59494</td>
<td>0.59493</td>
<td>0.59484</td>
<td>0.56095</td>
<td>0.59556 (0.001)</td>
</tr>
<tr>
<td>9</td>
<td>1.04554</td>
<td>1.04556</td>
<td>1.04372</td>
<td>1.0828</td>
<td>1.04630 (0.0013)</td>
</tr>
<tr>
<td>10</td>
<td>1.24228</td>
<td>1.24232</td>
<td>1.23715</td>
<td>1.36330</td>
<td>1.24298 (0.0014)</td>
</tr>
<tr>
<td>11</td>
<td>0.99428</td>
<td>0.99432</td>
<td>0.98663</td>
<td>1.11645</td>
<td>0.99432 (0.0015)</td>
</tr>
<tr>
<td>11.9999</td>
<td>0.48476</td>
<td>0.48599</td>
<td>0.48311</td>
<td>0.51978</td>
<td>0.48478 (0.0012)</td>
</tr>
<tr>
<td>12</td>
<td>0.48354</td>
<td>0.48355</td>
<td>0.47825</td>
<td>0.51834</td>
<td>–</td>
</tr>
<tr>
<td>CPU</td>
<td>120 sec.</td>
<td>124 sec.</td>
<td>770 sec.</td>
<td>161 sec.</td>
<td>340 sec.</td>
</tr>
</tbody>
</table>

The two barriers assume small values, i.e. $L = 4$, $U = 12$. The underlying asset price is $S_0$, the strike price is $K = 8$ and the other parameters are: $\sigma = 0.25$, $r = 0.05$, $T = 0.5$. Implicit-upwind scheme is faster than the other schemes but it is not the most accurate. However, the Crank-Nicolson variant scheme guarantees second-order accurate solution. The choice of efficient scheme depends on the aims: accuracy or computational speed.
Efficient and Accurate Schemes. Summary

★ We have examined various types of finite difference schemes for the Black-Scholes equation with discontinuous boundary conditions in case of discrete double barrier options.

★ We have proven that often frequently used schemes in computational finance such as the Crank-Nicolson scheme are inaccurate because these schemes could not satisfy all the financial requirements of the option contract.

★ Thus, a high accurate scheme is not necessarily the best one.

★ The choice of an accurate finite difference scheme depends strongly on the values of the parameters involved in the respective Black-Scholes equation for the option.

★ For pricing discrete double barriers options we have proposed efficient finite difference schemes by imposing some extra sufficient conditions for obtaining positive, free of oscillations and convergent finite difference solution in the supremum norm that is most relevant in Finance.
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Conclusions

★ We could say that there is not a universal finite difference scheme for pricing discrete double barrier options.

★ The choice of the scheme depends of the parameters of the option contract and mainly of the properties of the respective payoff function such as smoothness and continuity.

★ We have managed to propose variants of the well-known Crank-Nicolson scheme, implicit and semi-implicit schemes that satisfy all financial requirements of the option contract.

★ The schemes do not suffer of undesired spurious oscillations.

★ Unfortunately, for most path-dependent options there exist no explicit pricing formulas and the need for inventing finite difference scheme has never disappeared.

★ However, in this thesis we have demonstrated that in the discontinuous payoff option pricing the application of the finite difference schemes should carefully be done.
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With each change in asset price, instead of the absolute change as a measure, it is more appropriate to associate another useful indicator that is the return.

This quantity is defined to be the relative change in the price, i.e. $dS/S$. Let at time $t$ the asset price is $S$ and in a small subsequent time interval $dt$, $S$ changes to $S + dS$.

The main idea to model the corresponding return on the asset is to decompose this return into two parts according to anticipation: Predictable, deterministic and anticipated return akin to the return on money that are invested in a risk-free bank and random change in the asset price in response to external effects, such as unexpected news.
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The Black-Scholes Model

★ The first gives a contribution $\mu \, dt$ and the second $\sigma \, dX$. Putting together, we obtain the stochastic differential equation

$$\frac{dS}{S} = \mu \, dt + \sigma \, dX \quad (5)$$

where $\mu$ is a measure of the average rate growth of the asset price, also known as the drift, and $\sigma$ is a number called the volatility, which measures the standard deviation of the returns. Both parameters are constants.

★ The term $dX$ contains the randomness of asset prices and is known as a Wiener process.

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- The Black-Scholes model implies more about the behaviour of the price of a stock option, i.e. the stock’s price at time $T$, given its price today, is lognormally distributed.

- By Itô’s lemma, [71], an equivalent expression of equation (5) is obtained:

$$d \ln S = \left( \mu - \frac{1}{2} \sigma^2 \right) dt + \sigma dX \quad (6)$$

- Because $\mu$ and $\sigma$ are constant, this equation indicates that $\ln S$ follows a generalized Wiener process with drift rate $\mu - \sigma^2/2$ and constant variance rate $\sigma^2$.

- The change in $\ln S$ between time zero and some future time, $T$, is therefore normally distributed with mean $\left( \mu - \frac{1}{2} \sigma^2 \right) T$ and variance $\sigma^2 T$. This means that

$$\ln S_T \sim \phi \left[ \ln S_0 + \left( \mu - \frac{1}{2} \sigma^2 \right) T, \sigma \sqrt{T} \right] \quad (7)$$

where $\ln S_T$ is the stock price at a future time $T$ and $S_0$ is the stock price at time zero.
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where $\ln S_T$ is the stock price at a future time $T$ and $S_0$ is the stock price at time zero.
The Black-Scholes Analysis

- The Black-Scholes model implies more about the behaviour of the price of a stock option, i.e. the stock’s price at time $T$, given its price today, is lognormally distributed.

- By Itô’s lemma, [71], an equivalent expression of equation (5) is obtained:

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Semi-analytical Approaches

★ In the thesis we have explored both numerical and semi-analytical approaches using a model structure for discrete double barrier knock-out options that includes \( m \)-number independent normally distributed random variables where \( m \) is the number of the monitoring dates.

★ We present an analytical formula for the option value which represents a \( m \)-dimensional definite integral with limits including the two barriers and the strike price and we propose a fast and accurate algorithm for its valuation.

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The Black-Scholes Valuation Formula

★ The Black-Scholes model for determining the behaviour of the stock price turns out to be fundamental in option pricing, [6]. For constant interest rate and volatility the famous Black-Scholes formula gives an explicit formula for the value of European options on a non-dividend paying stock, [60].

★ But in addition to employing the simple asset price model, Fisher Black and Myron Scholes imposed several simplifying assumptions about the options market, [6].

✔ The following formula for the value of a European call option at time $t$ and asset price $S$ has been derived:

$$C(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2)$$  (8)

$$d_1 = \frac{\log(S/K) + (r + \frac{1}{2} \sigma^2)(T-t)}{\sigma \sqrt{T-t}}$$  (9)

$$d_2 = \frac{\log(S/K) + (r - \frac{1}{2} \sigma^2)(T-t)}{\sigma \sqrt{T-t}} = d_1 - \sigma \sqrt{T-t}$$  (10)

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Black-Scholes Surface for a European Call

Figure: 4. Black-Scholes surface for a European call option.
Monte Carlo Simulations

★ Monte Carlo method simulates the random movement of the asset prices and provides a probabilistic solution to the option pricing model, [40].

★ The idea is to generate a large number of sample paths of the process $S_T$ of the stock price over the interval $[0, T]$.

★ For each of these sample paths we compute the value of the function of the path whose expectation we wish to evaluate, and then to average these values over the sample paths.

✔ When European options are priced we need to estimate the expected value of the discounted payoff of the option at the maturity date $T$:

$$ V(S_0, 0) = e^{-rT} = e^{-rT} \hat{E}(V(S_T, T)) \tag{11} $$

✔ A constant risk-free rate is assumed, and the expectation $\hat{E}(\cdot)$ is taken with respect to a risk-neutral measure, i.e. the drift $\mu$ for the asset price must be replaced by the risk-free rate $r$. 
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Monte Carlo for Standard Options

★ The principle of the method is simple: if we denote with $K$ the strike price using the expected value of the discounted payoff of the option at the maturity date $T$, we should generate the option payoffs according to the expression

$$\max \left[ 0, \ S(0) e^{\left( r - \frac{1}{2} \sigma^2 \right) T + \sigma \epsilon \sqrt{T} } - K \right]$$

★ The computation of the sample average is straightforward;
★ The only difficulty is in quantifying and controlling the error.
✔ A large number of simulation runs are generally required in order to achieve a desired level of accuracy.
✔ To reduce the ‘error’ by a factor of 10 requires a hundredfold increase in the sample size.
★ This is a severe limitation that typically makes it impossible to get very high accuracy from a Monte Carlo approximation if it is applied in this form (also known as crude Monte Carlo).
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Monte Carlo for Exotics

★ Path-dependent options could be priced using crude Monte Carlo but a desired level of accuracy is not achieved quickly.

✔ Numerous variance reduction techniques such as antithetic variates, control variates, conditioning and importance sampling are used to be improved the Monte Carlo method, [8].

★ Unfortunately, as in case of a truncated payoff call option, antithetic sampling for discrete double barrier options is not guaranteed to work because the payoff is not a monotonic function of the stock price, see P. Brandimarte, [8].

★ In contrast to discretely sampled average price Asian call options, i.e. with arithmetic average $\hat{S} = \frac{1}{n} \sum_{i=1}^{n} S(t_i)$ in the payoff function $\max(\hat{S} - K, 0)$, for discrete double barrier options it is difficult to be found a control variate, Higham, [91].

✔ Pricing down-and-in call option with a discretely monitored barrier using an importance sampling technique and a conditional Monte Carlo is presented by Glasserman, [7].
Binomial and Trinomial Trees Difficulties for Barriers

★ In absence of a valuation formula for non-standard options or American options with dividends, binomial and trinomial trees are the simplest means for pricing, see Hingham, [89].

★ In a binomial method, given an initial asset price, we first build a tree of possible values of asset prices and their probabilities. Using the tree the possible asset prices at expiry are determined and all the probabilities.

★ The possible values of the option at expiry are calculated, and by working back down the tree using a mathematical relation of the tree nodes, the option is valued.

★ Remark: We have assumed that the payoff function is determined only by the value of the underlying asset at the expiry. This is not the case, for example, for a path-dependent options such as barrier options, lookback and Asian ones.

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★ For the possible values of the asset price the *trinomial tree* has one more state (degree of freedom) than the binomial tree.

★ The trinomial method has proven to be more useful and adaptable for many derivative applications.

★ There is a very close connection of *trinomial trees* and the explicit finite difference schemes, see Hull, [60].

✔ Including barrier constraints causes difficulties of adjusting the tree, see Kwok, [14]. One decision is the algorithm of Tian-Boyle both for the continuous and the discrete case, [23].

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✔ Recently, options with discontinuous payoffs such as discrete barrier options are valuated using a more complicated lattice model for the asset price such as special types of *trinomial trees with adaptive mesh mechanism* around the nonlinear payoff region of exercise price at maturity, see Shea, [17].
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- Recently, options with discontinuous payoffs such as discrete barrier options are valuated using a more complicated lattice model for the asset price such as special types of *trinomial trees with adaptive mesh mechanism* around the nonlinear payoff region of exercise price at maturity, see Shea, [17].
We have applied the Crank-Nicolson finite difference scheme to three examples in order to be explored in consecutive steps and better understand the difficulties of the valuation problem in case of discrete double barrier options.

Firstly, we explore the heat equation with zero boundary conditions and make a comparison of the analytical solution and the Crank-Nicolson scheme finite difference solution: we compare experimentally the error between the exponential and amplification terms of the two solutions.

Second, we explore a more complicated example, i.e. the initial-boundary value problem with discontinuous initial conditions defined by Stikwerda in [79].

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★ We make a convergent analysis by examining the numerical error both in the supremum and the $L^2$-norm.

★ To achieve convergence of the numerical solution in the supremum norm, we impose on the finite difference scheme some extra sufficient conditions for stability such as the discrete version of the maximum principle.

★ Third, we explore the Black-Scholes equation in case of discrete double barrier options that is the main problem and propose accurate finite difference schemes for it.

✓ In fact, our finite difference suggestions are variants of the standard Crank-Nicolson scheme, the full-implicit and semi-implicit schemes but differ with some extra imposed sufficient conditions and thus the schemes satisfy all the financial requirements of the option contract such as positivity and do not suffer of undesired spurious oscillations.
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Fourier Analysis. Convergence

- The Black-Scholes equation could be used in option pricing for valuation of a wide range of options.
- By appropriate changing of the variables, the Black-Scholes equation could be transformed into a heat equation.
- The advantage is that the obtained equation has constant coefficients and is well studied via the Fourier Analysis.
- Moreover, in the discrete case, applying the Fourier’s method and comparing the exact solution and the finite difference solution permits observing the convergence and accuracy of the applied scheme and thus its efficiency, Tveito, [75].

✔ On the other side, Fourier’s method cannot handle nonlinear equations or linear but with variable coefficients such as the Black-Scholes equation. Nevertheless, there is similarity between Fourier’s method and the eigenvalue/eigenvector problem for linear system of ordinary differential equations.
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★ The numerical solution of the respective *discrete equation* applying the Crank-Nicolson scheme should exhibit convergent properties similar to the analytical solution.

✔ In other words, the respective numerical solution should also damp high-frequency terms of the initial function.

★ We have demonstrate that the error of the Crank-Nicolson numerical solution does not decrease when measured in the *supremum norm*, but it does decrease in the $L^2$-norm.

★ The $L^2$-error norm is usually defined in the following way:

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\|u(S, T) - V(S_j, T)\|_{L^2} \approx \sqrt{\Delta S \sum_{l=0}^{M} \|u(S_j, T) - V(S_j, T)\|}
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Starting from the initial Black-Scholes equation, using finite difference approach the problem is reduced to solving a linear system of the type $AX = B$.

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The particular derivative depends on the boundary conditions that are used in the Black-Scholes equation, [60].

Pricing non-standard options with payoff with discontinuities, reflects in discontinuities in the boundary conditions.

Such options are binary and discrete barrier options.

The major part of the thesis is devoted to the application of the finite difference schemes to the Black-Scholes equation that has non-smooth or discontinuous boundary conditions in case of discrete double barrier options.

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In such cases the financial provisions of the contract can affect strongly the reliability of the numerical solution by reflecting the terminal conditions of the respective Black-Scholes partial differential equation for the particular derivative.

The discontinuity will be renewed at every monitoring clause date in case of discrete double barrier options. So the situation gets worse when we have monitoring in discrete moments.

The oscillations derive from an inaccurate approximation of the very sharp gradient produced by the knock-out clause in case of discrete double barrier options, generating an error that is damped out very slowly.

This problem could be avoided if the time-step becomes prohibitively small, see Tagliani, [18], and also Tavella, [19].

Thus, the direct application of the Crank-Nicolson method is not efficient and inventing suitable finite difference schemes to solve accurately the Black-Scholes equation is required.
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Crank-Nicolson Scheme for Discrete Double Barrier Options

Figure: 5. Option pricing just before the last monitoring date $t_F = 12$. $L = 90, K = 100, U = 110, r = 0.05, \sigma = 0.2, T = 1, \Delta S = 0.1, \Delta t = 0.01.$
Discrete Monitored Barrier Options: by Finite Difference Schemes

Aldo Tagliani, Faculty of Economics
Ph D Mariyan Milev, Faculty of Mathematics

The Main Problem
Options with Discontinuous Payoff Function
Option Valuation Problem. Postulation.

Difficulties in Pricing of Discrete Double Barrier Options
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Works or Not for Options with Discontinuous Payoffs?

Figure: 6. Payoff diagram of a barrier option with discrete monitoring.
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- We propose variants of the standard Crank-Nicolson scheme, implicit and semi-implicit schemes that differ with some extra imposed sufficient conditions such as the discrete version of the maximum principle, see Tagliani [18].

- The spectrum of the iteration matrix of the finite difference scheme contains uniquely positive eigenvalues.

- Thus, the finite difference schemes satisfy all the financial requirements of the option contract such as positivity and do not suffer of undesired spurious oscillations.

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Discrete Maximum Principle. Application

★ It is therefore reasonable that the numerical scheme possesses a similar property.

★ Unfortunately, due to variable coefficients in option pricing models, in general, the preservation of the discrete maximum principle can depend on the values of the model parameters such as interest rate and volatility, independently of the discretization steps. For example, discrete barrier options.

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$$\max_{S \in [0, S_{\text{max}}]} |V(S, t_1)| \geq \max_{S \in [0, S_{\text{max}}]} |V(S, t_2)|, \quad t_1 \leq t_2 \quad (12)$$

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M-matrices

• The necessity of studying *M-matrices* has emerged from the numerous application of matrices with such structure.

• It turns out that such kind of matrices have a variety of useful properties that makes them a valuable tool in Numerical analysis, Biomathematics and Finance.

• In general many parabolic equations with *non-smooth or discontinuous* boundary conditions are treated better when *M-matrices* are used as an *iteration matrix* because:
  ✔ The finite difference schemes are *free of oscillations*
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Definition and Structure of M-matrices

- All the discussed matrices are real. We will present the most frequently used in practice definitions of *M-matrices*, [59]:

  ✔ **Definition 1:** A matrix \( A = (a_{ij}) \) is called an M-matrix if \( a_{ij} \leq 0 \) whenever \( i \neq j \) and all principal minors of \( A \) are positive.

  ✔ **Definition 2:** (Fan) A matrix \( A = (a_{ij}) \) with *nonpositive* off-diagonal elements is an M-matrix if and only if \( A \) is nonsingular and \( A^{-1} \) is *nonnegative*.

  ✔ **Definition 3:** Any matrix \( A \) of the form \( A = sI - B \), with \( s > 0 \) and \( B \geq 0 \) for which \( s > S(B) \), where \( S(B) \) is the spectral radius of the nonnegative matrix \( B \), is called an M-matrix. For \( s = S(B) \), the matrix \( A \) is a singular M-matrix.

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Classification of M-matrices

\[ Z^{n \times n} = \{ A = (a_{ij}) : a_{ij} \leq 0, \ i \neq j \} \]

\[ A = \begin{pmatrix} -1 & -2 \\ 0 & 1 \end{pmatrix} \]

\[ A = \begin{pmatrix} 1 & -3 \\ -1 & 1 \end{pmatrix} \]

\[ A = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix} \]

\[ A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \]

\[ A = \begin{pmatrix} 4 & -1 & -1 \\ -1 & 4 & 1/4 \\ -1 & -1 & 4 \end{pmatrix} \]
Jacobi and M-matrices. Positivity

★ **Definition 1:** Tridiagonal real square matrix with off-diagonal elements that have the same sign is called a *Jacobi matrix*.

✔ **Theorem 1:** An irreducible tridiagonal matrix $A \in L^{n \times n}$ is a non-singular (singular) *M-matrix* if and only if the smallest eigenvalue of $A$ is *positive* (zero).

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Discrete Monitored Barrier Options: by Finite Difference Schemes

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The Main Problem

Options with Discontinuous Payoff Function

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> Thus, using the $L^{n \times n}$ structure of M-matrices, the constants $a$ and $b$ are chosen by the following criteria:

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★ Under this condition $P$ is also an irreducible matrix.

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★ and all diagonal elements of the matrix $N$ are positive if

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$$a = b = -\frac{r}{16 \sigma^2}, \quad (15)$$

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Discrete Monitored Barrier Options: by Finite Difference Schemes

Aldo Tagliani, Faculty of Economics
Ph D Mariyan Milev, Faculty of Mathematics

The Main Problem

Options with Discontinuous Payoff Function

Option Valuation Problem. Postulation.

Difficulties in Pricing of Discrete Double Barrier Options

Crank-Nicolson Classic Scheme in Finance.

Works or Not for Options with Discontinuous Payoffs?
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By combining $P^{-1} > 0$ and $N \geq 0$, it follows that the iteration matrix is positive, i.e. $P^{-1}N > 0$ and the numerical solution $V_{n+1} = (P^{-1}N)V_n = (P^{-1}N)^nV_0$ is also positive since $V_0 \geq 0$.

$P$ is a strongly row diagonally dominant matrix and then

$$||P^{-1}||_\infty \leq \left( \frac{1}{\Delta t} + \frac{r}{2} \right)^{-1}, \quad \text{Windish, [87]}$$

Using the norms $||P^{-1}||_\infty$ and $||N||_\infty = \frac{1}{\Delta t} - \frac{r}{2}$, we verify that the numerical solution of the scheme satisfies the discrete maximum principle and this is sufficient for its stability:

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✔ Then the absence of spurious oscillations requires restriction on the time step $\Delta t$ uniquely while restrictions on the financial parameters $r$ and $\sigma$ are not required. Thus, absence of spurious oscillations is guaranteed if the time step verifies

$$\Delta t < \Delta t_1 = \frac{1}{r(\frac{1}{2} - 4b) + (\sigma M)^2}$$

★ For large $M$ values, we notice that $\Delta t_1 \approx \frac{1}{(\sigma M)^2} =: \tau_d$, i.e. the characteristic grid diffusion time of Tavella, [19].
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Crank-Nicolson Scheme Analysis when $\sigma^2 > r$.

Defining Eigenvalues of the Iteration Matrix

✔ The analysis permits to be examined only the case $\sigma^2 > r$.

\[
P = \frac{1}{\Delta t} I + C \quad \text{and} \quad N = \frac{1}{\Delta t} I - C , \text{ where} \quad (16)
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C = \text{tridiag} \left\{ \frac{r S_j}{4 \Delta S} - \left( \frac{\sigma}{2 \Delta S} \right)^2; \frac{1}{2} \left( \frac{\sigma S_j}{\Delta S} \right)^2 + \frac{r}{2}; -\frac{r}{4 \Delta S} - \left( \frac{\sigma}{2 \Delta S} \right)^2 \right\}
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Negative Eigenvalues of the Iteration Matrix That are Close to \(-1\). Factors For Oscillations

\begin{itemize}
  \item ✔ Diminishing the space step \(\Delta S\), i.e. \(M \rightarrow \infty\), then the spectrum \(\rho(C) := \max(\lambda_i(C)) \rightarrow \infty\) and \(\lambda_i(P^{-1}N) = \frac{1-\Delta t \lambda_i(C)}{1+\Delta t \lambda_i(C)} \rightarrow -1\).

  \item ✔ As a consequence the numerical solution \(V_{n+1}\) oscillates:

\[ V_{n+1} = (P^{-1}N)^n V_0 = (P^{-1}N)^n \sum_{i=1}^{M} d_i v_i = \sum_{i=1}^{M} d_i (P^{-1}N)^n v_i = \sum_{i=1}^{M} d_i (\lambda_i)^n v_i \]

\item ✔ Tavella describes this Crank-Nicolson phenomena as excitation of the eigenvalues of the finite difference matrix. He introduces the so called characteristic grid diffusion time \(\tau_d := \frac{\Delta S^2}{(\sigma S)^2}\) so that when \(\Delta t \gg \tau_d\) oscillations may occur, [19].
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★ To the $M$ distinct eigenvalues $\lambda_i(P^{-1}N)$ could be associated $M$-number linearly independent eigenvectors $v_i$ that can be used as a basis for the $M$-dimensional space of the payoff $V(0)$ that is $V(0) = \sum_{i=1}^{M} d_i v_i$, where the $d_i$ are proper weights.

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Negative Eigenvalues of the Iteration Matrix That are Close to $-1$. Factors For Oscillations

- Diminishing the space step $\Delta S$, i.e. $M \to \infty$, then the spectrum
  \[ \rho(C) = \max(\lambda_i(C)) \to \infty \]  
  \[ \lambda_i(P^{-1}N) = \frac{1-\Delta t \lambda_i(C)}{1+\Delta t \lambda_i(C)} \to -1. \]

- To the $M$ distinct eigenvalues $\lambda_i(P^{-1}N)$ could be associated $M$-number linearly independent eigenvectors $v_i$ that can be used as a basis for the $M$-dimensional space of the payoff $V(0)$ that is $V(0) = \sum_{i=1}^{M} d_i v_i$, where the $d_i$ are proper weights.

- As a consequence the numerical solution $V_{n+1}$ oscillates:
  \[ V_{n+1} = (P^{-1}N)^n V_0 = (P^{-1}N)^n \sum_{i=1}^{M} d_i v_i = \sum_{i=1}^{M} d_i (P^{-1}N)^n v_i = \sum_{i=1}^{M} d_i (\lambda_i)^n v_i \]

- Tavella describes this Crank-Nicolson phenomena as excitation of the eigenvalues of the finite difference matrix. He introduces the so called characteristic grid diffusion time
  \[ \tau_d = \frac{\Delta S^2}{(\sigma S)^2} \]  
  so that when $\Delta t \gg \tau_d$ oscillations may occur, [19].
Discrete Monitored Barrier Options: 
by Finite Difference Schemes

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Option Valuation Problem. Postulation.
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