Copula-based Martingale Processes and Financial Prices Dynamics

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Outline

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2. Copula Functions and Markov processes
3. Models with (in)dependent increments
4. Martingale Markov processes
5. Copula characterization of bivariate Markov processes
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Pricing Problem

Many correlation products are based on prices of a set of underlying assets observed at different dates.

Cross-section Compatibility

The price has to be consistent with those of the univariate assets at any given time.

Temporal Compatibility

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Dependence in Finance

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Assume an $m$-dimensional copula $A$ and an $n$-dimensional copula $B$: the $\star$ operator is defined as

$$A \star B(u_1, u_2, \ldots, u_{m+n-1}) = \int_0^1 \frac{\partial A(u_1, \ldots, u_{m-1}, t)}{\partial t} \frac{\partial B(t, u_{m+1}, \ldots, u_{m+n-1})}{\partial t} \, dt$$

Theorem (Darsow, Nguyen and Olsen, 1992)

A real valued stochastic process $X_t$ is a Markov process if and only if, for all positive integers $n$ and all $t_1 < t_2 < \ldots < t_n$

$$C_{t_1 \ldots t_n} = C_{t_1, t_2} \star C_{t_2, t_3} \star \ldots \star C_{t_{n-1}, t_n}$$

where $C_{t_1 \ldots t_n}$ is the copula of $(X_{t_1}, \ldots, X_{t_n})$ and $C_{t_{k-1}, t_k}$ is the copula of $(X_{t_{k-1}}, X_{t_k})$. 
Assume an $m$-dimensional copula $A$ and an $n$-dimensional copula $B$: the $*$ operator is defined as

$$A \star B(u_1, u_2, ..., u_{m+n-1}) \equiv \int_0^u \frac{\partial A(u_1, ..., u_{m-1}, t)}{\partial t} \frac{\partial B(t, u_{m+1}, ..., u_{m+n-1})}{\partial t} \, dt$$

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Our approach: dependent increments

Proposition

Take three continuous distributions $H$, $F$, $G$ and $C(u, v)$ a bivariate copula function. Let $D_1 C(u, v) = \frac{\partial C(u, v)}{\partial u}$. Then the function

$$\hat{C}(u, v) \equiv \int_0^u D_1 C[w, F(G^{-1}(v) - H^{-1}(w))]dw$$

is a copula function iff

$$\int_0^1 D_1 C[w, F(z - H^{-1}(w))]dw \equiv H \ast F(z) = G(z)$$
In the context of random variable

Given \((Z, W)\) such that

- \(C(u, v)\) is the associated copula;
- \(H\) is the c.d.f. of \(Z\);
- \(F\) is the c.d.f. of \(W\), then

- \(\hat{C}(u, v)\) is the copula associated to \((Z, W + Z)\)
- \(H^C F\) is the c.d.f of \(W + Z\).
In the context of processes

- $F$ represents the probability distribution of increments of the process $X_t - X_s$, $H$ represents the distribution of $X_s$ and $H^C \ast F$ represents the distribution of $X_t$;
- $\hat{C}(u, v)$ is the copula $C_{s,t}$ associated to $(X_s, X_t)$;

Lévy processes:
- $C(u, v) = uv$ ⇒ the operator is the convolution;
- $F$ depends only on $t - s$ (stationary increments).
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An algorithm to construct discrete time Markov processes

- Denote $X_{t_{i-1}}$ the process at time $t_{i-1}$ and $H_{i-1}$ the corresponding c.d.f.;
- Denote $Y_{t_i}$ the random increment of the process in the period $[t_{i-1}, t_i]$ and $F_i$ the corresponding c.d.f.;
- Denote $C_i(u, v)$ the copula associated to $(X_{t_{i-1}}, Y_{t_i})$.

1. Start with the probability distribution $H_1$ of $X_{t_1};$
2. Compute $H_2(z) = H_1 \ast^C F_2$
   
   $$C_{t_1, t_2}(u, v) = \int_0^u D_1 C_2[w, F_2(H_2^{-1}(v) - H_1^{-1}(w))]dw;$$
3. Go back to step 2, and using $F_3$ and $H_2$ compute $H_3$...
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Martingale Markov processes

Copula characterization

Let $X$ be a Markov process and $\Delta_{t-s}X_s = X_t - X_s$. Denote with $C^{t,s}$ the copula associated to $(X_s, \Delta_{t-s}X_s)$ and $F_{s,t}$ the c.d.f. of $\Delta_{t-s}X_s$. $X$ is a martingale iff:

1. $F_{s,t}$ has finite mean for every $s, t$;
2. for every $s, t$, $\int_{0}^{1} F_{s,t}^{-1}(v) d(D_1 C^{t,s}(u, v)) = 0$, $\forall u \in [0, 1]$. 
**Proposition**

If the law of $F_{s,t}$ is symmetric and

$$C^{s,t}(u, v) + C^{s,t}(u, 1 - v) = u \quad (1)$$

then $X$ is a martingale.

Let $A(u, v)$ be a bivariate copula and $\hat{A}(u, v) \equiv u - A(u, 1 - v)$.

$$C(u, v) \equiv 0.5A(u, v) + 0.5\hat{A}(u, v)$$

is a copula that satisfies (1).
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Let $m, n \geq 2$ and $A$ and $B$ be, respectively, $m$- and $n$-dimensional copulas. If

- $A_{1,\ldots,m|m-1,m}(u_1,\ldots,u_{m-2},\xi,\eta) = \frac{\partial^2 A(u_1,\ldots,u_{m-2},\xi,\eta)}{\partial \xi \partial \eta}$;
- $B_{1,\ldots,n|1,2}(\xi,\eta,u_3,\ldots,u_n) = \frac{\partial^2 B(\xi,\eta,u_3,\ldots,u_n)}{\partial \xi \partial \eta}$;
- $A(1,\ldots,1,\xi,\eta) = B(\xi,\eta,1,\ldots,1) = C(\xi,\eta)$

Then

$$A \star^2 B(u_1,\ldots,u_{m+n-2}) = \int_0^{u_{m-1}} \int_0^{u_m} A_{1,\ldots,m|m-1,m}(u_1,\ldots,u_{m-2},\xi,\eta)B_{1,\ldots,n|1,2}(\xi,\eta,u_3,\ldots,u_n) dC(\xi,\eta)$$
Let \((X, Y)\) be an \(\mathbb{R}^2\)-valued stochastic process and 
\(C_{t_1,t_2,\ldots,t_n}(u_1, v_1, \ldots, u_n, v_n)\) denote the \(2n\)-dimensional copulas associated to \((X_{t_1}, Y_{t_1}, X_{t_2}, Y_{t_2}, \ldots, X_{t_n}, Y_{t_n})\).

An \(\mathbb{R}^2\)-valued stochastic process \((X, Y)\) is a Markov process if and only if for all \(t_i, i = 1, \ldots, n\), such that \(t_1 < \ldots < t_n\)

\[C_{t_1,t_2,\ldots,t_n} = C_{t_1,t_2} \star^2 C_{t_2,t_3} \star^2 \cdots \star^2 C_{t_{n-1},t_n}.\]
Martingale condition

The Markov process $(X, Y)$ is a martingale with respect to the filtration $\mathcal{F}^{X,Y}$ iff:

1. $F_{\Delta_h X_t}$ and $F_{\Delta_h Y_t}$ have finite mean for every $t$;
2. for every $t, h$,

$$\int_0^1 F_{\Delta_h X_t}^{-1}(w) d \left( D_{1,2} C^{t+h}(u, v, w, 1) \right) = 0, \quad \forall u, v \in [0, 1]$$

and

$$\int_0^1 F_{\Delta_h Y_t}^{-1}(w) d \left( D_{1,2} C^{t+h}(u, v, 1, w) \right) = 0, \quad \forall u, v \in [0, 1].$$
Granger causality

In the setting of Markov processes, $Y$ does not Granger cause $X$ if

$$\mathbb{P}[X_{t+h} \leq x | X_t, Y_t] = \mathbb{P}[X_{t+h} \leq x | X_t].$$

⇓

If $X$ is an $\mathcal{F}^{X,Y}$ Markov process, it is an $\mathcal{F}^X$ Markov process as well.

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Granger causality and copulas

Copula characterization

$X$ and $Y$ do not Granger cause each other iff

$$C_{t,t+h}(u, v, u', 1) = C_{Y,t,X_t} \ast C_{X,t,X_{t+h}}(v, u, u')$$

$$C_{t,t+h}(u, v, 1, v') = C_{X,t,Y_t} \ast C_{Y,t,Y_{t+h}}(u, v, v')$$
Hierarchical representation

If $X$ and $Y$ do not Granger cause each other the copula function $C_{t_1, t_2, \ldots, t_n}$ can be in a Hierarchical form

$$C_{t_1, \ldots, t_n}(u_1, v_1, \ldots, u_n, v_n) = C(G(u_1, \ldots, u_n), H(v_1, \ldots, v_n))$$

where the notation means that

$$G(u_1, \ldots, u_n) = C_{t_1, t_2} \ast \cdots \ast C_{t_{n-1}, t_n}(u_1, \ldots, u_n)$$
Barrier Altiplanos, Cherubini-Romagnoli (2008)

- Assume a note paying a set of coupons in \( k = 1, 2, \ldots, T \)
- Coupons are digital options indexed to a set of assets \( i = 1, 2, \ldots, n \)
- In each period \( k \) the price of assets is monitored at a set of dates \( j = 1, 2, \ldots, m_k \)
- Coupons are paid iff all the assets are above a barrier at all the reset periods
- The value of each coupons is exposed to \( n \times m_k \) risk factors and to their dependence structure.

We propose a pricing model based on Independent increments processes, Markov assumption and no Granger causality.
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Copula characterization of bivariate Markov processes
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For Further Reading

VaR Aggregation, Cherubini-Mulinacci-Romagnoli (2009)

VaR Aggregation: temporal dimension

Let $X_{i-1}$, $Y_i$ be two random variables with continuous c.d.f. $F_{X_{i-1}}$, $F_{Y_i}$ and linked by temporal copula function $C^i$, our task is to determine VaR of $X_i$:

$$\int_0^1 D_1 C^i (w, F_{Y_i}(\text{VaR}_c(X_i) - \text{VaR}_{1-w}(X_{i-1}))) \, dw = 1 - c$$

where $\text{VaR}_c(X_i) = F_{X_i}^{-1}(1 - c)$
In this application we assume that $Y_{t_i}$ (with c.d.f. $F_i$) is the percentage of losses in every time period $[t_{i-1}, t_i]$, $X_{t_{i-1}}$ (with c.d.f. $H_i$) gives the cumulated relative losses at time $t_{i-1}$ and $C_i(x, y)$ is the associated copula function.

Then, the law of the dynamics of cumulated losses is recovered by iteratively computing the convolution-like relationship

$$H_i(z) = \int_0^1 D_i C_i[w, F_i(z - H_{i-1}^{-1}(w))] dw, \quad i = 2, \ldots$$

The constrains $0 \leq X_{t_i} \leq 1$ and $0 \leq Y_{t_i} \leq 1$ have to be imposed.
Constrained Problem

Proposition

Let $X$ and $Y$ be two continuous random variables. Let $[0, \alpha]$ and $[0, \beta]$, respectively, their support, $F_X$ and $F_Y$ their distribution and $C$ the copula function associated to the random vector $(X, Y)$. If $\gamma \geq \alpha \vee \beta$, $\mathbb{P}(X + Y \leq \gamma) = 1$ if and only if

$$D_1 C_{X,Y}(w, F_Y(\gamma - F_X^{-1}(w))) = 1, \quad \forall w \in [0,1] \quad \text{a.e.} \quad (2)$$

If $\alpha + \beta \leq \gamma$, (2) is satisfied for every copula function $C$ and all marginal distributions $F_X$ and $F_Y$. 
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If $\alpha + \beta \leq \gamma$, (2) is satisfied for every copula function $C$ and all marginal distributions $F_X$ and $F_Y$. 

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Copula-based Martingale Processes
Constrained Problem

We are interested in studying the equation

\[ D_1 C(u, v) = 1. \]  

(3)

Let \( S_C = \{(u, v) \in [0, 1]^2 : D_1 C(u, v) = 1\} \).

Proposition

\((u, 1) \in S_C \) for almost all \( u \in [0, 1] \).

Proposition

If there exists \( \epsilon > 0 \) such that \( \forall u \in (\epsilon, 1], (u, v) \in S_C \) iff \( v = 1 \), then (2) holds if and only if \( \alpha + \beta \leq \gamma \).
Constrained Problem: some example

Archimedean Copulas

Let $C(u, v)$ be an Archimedean copula with generator $\phi$. $S_C = \{(u, 1) : u \in [0, 1]\}$ if and only if $\phi'$ is invertible. If $\phi'$ is not invertible in an open interval $(a, b)$ then

\[
\{(u, v) : u \in (a, b), v \in [\phi^{-1}(\phi(a) - \phi(u)), 1]\} \subset S_C.
\]

Andersen, L. and J. Sidenius (2004), "Extensions to the Gaussian copula: random recovery and random factor loading", Journal of Credit Risk, 1, 29-70

For Further Reading II


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For Further Reading


Gregory, J., and J.P. Laurent (2003), "Basket default swaps, CDOs and factor copulas", ISFA Actuarial School and BNP Parisbas, Working paper
For Further Reading IV


For Further Reading V
