Local volatility surfaces for illiquid currency pairs

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Outline

1 Motivation: illiquid cross currency options
   - 1D Black-Scholes PDE
   - The 2D PDE for the price

2 Local volatility calibration: choice of variables
   - Algorithm
   - Results

3 The 2D PDE for the price: boundary conditions

4 Conclusion and perspectives
In practice FOREX options are quoted on the market such that we know the price of many of them for major currency pairs: EUR/USD, JPY/USD, GBP/USD, EUR/GBP, etc.

What about a firm wanting to hedge the risk of Japan yen (JPY) versus e.g., New Zeeland $ (NZD) ?

They need an JPY/NZD option but only available are $S_1 = JPY/USD$ and $S_2 = NZD/USD$. What is the price of a cross-currency JPY/NZD option ?
Price of a call option: 1D Black & Scholes PDE

Suppose NZD/JPY cross-rate $S_t$ follows the SDE

$$dS_t/S_t = (r - q)dt + \sigma dW_t$$

Here $r =$ interest rate for the first currency (JPY), $q =$ interest rate for the second currency (NZD), $\sigma = \sigma_{JPY/NZD}$ is the volatility. The price $C$ follows the Black-Scholes PDE

$$\partial_t C + (r - q)S \partial_S C + \frac{\sigma^2 S^2}{2} \partial_{SS} C - rC = 0$$

$$C(S, t = T) = (S_T - K)_+.$$  

Price today is $C(S_0, t = 0)$. $\sigma_{JPY/NZD}$ is unknown and there are no options quoted.
Price of cross-currency options: 2D Black-Scholes

Price of a cross-currency option follows the 2D PDE:

\[
\begin{align*}
\partial_t c + S_1 \mu_1 \partial_{S_1} c + S_2 \mu_2 \partial_{S_2} c + \\
+ \frac{\sigma_1^2(S_1)^2}{2} \partial_{S_1} S_1 c + \frac{\sigma_2^2(S_2)^2}{2} \partial_{S_2} S_2 c + \rho_{1,2} S_1 S_2 \sigma_1 \sigma_2 \frac{\partial^2 c}{\partial S_1 \partial S_2} \\
- r c = 0
\end{align*}
\]

(4)

\[
c(T, S_1, S_2) = S_2 \left[ \frac{S_1}{S_2} - K \right]_+ = \left[ S_1 - KS_2 \right]_+. 
\]

(5)

\[
\mu_1 = r - r_1, \quad \mu_2 = r - r_2, \quad r, \rho_{1,2} \text{ are known.}
\]

Remark: constant \( \sigma_1 \) and \( \sigma_2 \): 2D PDE reduces to 1D PDE

To do:

- local volatility calibration problem from quoted option prices (on \( S_1 \) and \( S_2 \)) obtain the surfaces \( \sigma_1 \) and \( \sigma_2 \)
- solve the 2D PDE Remark: also possible by Monte-Carlo, should get same result.
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Local volatility calibration: choice of variables

Calibration: we have several prices quoted for different strikes $K_\ell$ and maturities $T_\ell : \mathcal{C}_{K_\ell, T_\ell}^{market}$. One needs to recover $\sigma = \sigma(S, t)$ such that $C(S_0, t = 0; K_\ell, r, T_\ell, \sigma(S, t)) = \mathcal{C}_{K_\ell, T_\ell}^{market}$ for all $\ell$.

$$\partial_t C^\ell + (r - q) S \partial_S C^\ell + \frac{\sigma^2 S^2}{2} \partial_{SS} C^\ell - r C^\ell = 0 \quad (6)$$

$$C^\ell(S, T_\ell) = (S - K_\ell)_+ \quad (7)$$

Procedure: automatic (because only an intermediary product) i.e. results robust with respect to user parameters, numerically stable w/r to input (even for hard cases e.g. $JPY/USD$), (fast), precise ($< 0.1\%$ in implied vol).
Calibration: what variables to chose

- Existing literature: all invert the mapping $\sigma \rightarrow C$.
- Coleman et Li, Verma: parametric setting
- L. Jiang et al., Achdou & Pirroneau: control framework (adjoints)
- Lagnado et al.: constraints

- Consider the mapping local to implied instantaneous variance $v = \sigma^2 \rightarrow \mathcal{V}(v) = w = (\sigma^I)^2$

1. asymptotics: if $\lim_{S \rightarrow 0/\infty} v(S, t) = v_{-/+}(t)$ then
   $\lim_{S \rightarrow 0/\infty} \mathcal{V}(v)(S, t) = \frac{1}{t} \int_0^t v_{-/+}(s)ds$, i.e. mapping is LINEAR

2. for $\sigma = \sigma(t)$ one can also prove numerical stability properties

- Idea: use this mapping for inversion.
Calibration: local to implied mapping

minimize the functional

\[ J^\epsilon(v) = \epsilon \| \nabla v \|_{L^2}^2 + \sum_{\ell=1}^{L} (\mathcal{V}(v)(K_\ell, T_\ell) - \mathcal{V}(v_0)(K_\ell, T_\ell))^2 \]  \hspace{1cm} (8)

Sequentially Quadratic Problem (SQP) (+ regularization, constraints)

\[ \min_p \left\{ J^\epsilon(v^k) + D_v J^\epsilon(v^k)(p) + \frac{1}{2} D_{vv} J^\epsilon(v^k)(p, p) \bigg| v^k + p \in \mathcal{K} \right\} \]  \hspace{1cm} (9)

Then one sets \( v^{k+1} = v^k + p^k \) (\( p^k \) is the solution of the QP).

Gauss-Newton style approximation for each QP: only need \( D_v \mathcal{V}(v) \).
Calibration: optimization algorithm

\[ D_v \mathcal{V}(v)(K_\ell, T_\ell) = 2\sqrt{\mathcal{V}(v)(K_\ell, T_\ell)} \frac{1}{\nu_{l,BS}} D_v C^\ell. \] (10)

The optimization algorithm: for \( D_v C \) use an adjoint

\[
\partial_t \chi + \partial_S ((r - q)S\chi) - \partial_{SS} \left( \frac{\nu S^2}{2} \chi \right) + r\chi = 0
\] (11)

\[ \chi(S, t = 0) = \delta_{S=S_0}. \] (12)

then

\[ D_v C^\ell(S_0, 0) = \frac{S^2}{2} (\partial_{SS} C^\ell) \chi. \] (13)

Parametric:

\[ \nu(S, t) = \sum \alpha_{ij} f_{ij}(S, t). \] (14)
Figure: Shape form.
Figure: Benchmark from Andersen et al., Coleman et al. for S&P 500 options.
Figure: Implied volatility surface for JPY/USD options dated March 18th 2008, courtesy of Thomson Reuters Financial Software.
Figure: Local volatility surface for JPY/USD options dated March 18th 2008, courtesy of Thomson Reuters Financial Software.
Figure: Local volatility surface for the South African Rand (ZAR) vs. USD options Nov. 2008, courtesy of Thomson Reuters Financial Software.
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$$\begin{align*}
\partial_t c + S_1 \mu_1 \partial_{S_1} c + S_2 \mu_2 \partial_{S_2} c + \\
\frac{\sigma_1^2(S_1)^2}{2} \partial_{S_1 S_1} c + \frac{\sigma_2^2(S_2)^2}{2} \partial_{S_2 S_2} c + \rho_{1,2} S_1 S_2 \sigma_1 \sigma_2 \frac{\partial^2 c}{\partial S_1 \partial S_2} \\
-rc = 0
\end{align*}$$

(15)

$$c(T, S_1, S_2) = S_2 [S_1/S_2 - K]_+ = [S_1 - KS_2]_+. \quad (16)$$

Yet to chose: the boundary conditions: we use Dirichlet B.C...
The 2D PDE for the price: boundary conditions

- (green) $S_1/S_2 << K$: $c(t, S_1, S_2) = 0$;
- (blue) $S_1/S_2 >> K$: use put-call parity
  $c(t, S_1, S_2) = S_1 e^{-r_1(T-t)} - KS_2 e^{-r_2(T-t)}$
- (red) $S_1, S_2$ both small/large: $\sigma_1, \sigma_2$ are undetermined: we take them constant and use the explicit Black-Scholes formula

**Figure:** The domain in $S_1/S_2$. 

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4. Conclusion and perspectives
• procedure tested on simple cases and approximations are ok; real-life cases are under implementation

• one may also look for a Dupire-like equation for a 2D continuum of option prices (cf. recent works by Pironneau et al.).

Refs: preprints: scholar google search ”Gabriel Turinici calibration” and http://hal.archives-ouvertes.fr;
• calibration local→ implied var.: to appear in J. Comp. Fi. (2009)

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