Discontinuous Galerkin Methods

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Lecture 2
An introduction to DG methods for elliptic problems

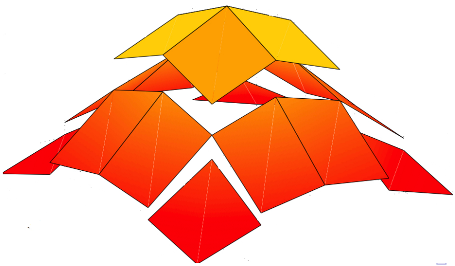
What are Discontinuous Galerkin methods?

Discontinuous Galerkin (DG) methods are a family of finite element methods for the approximation of partial differential equations

The idea

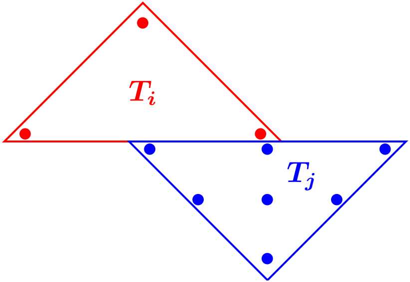
The discrete solution is seek in a discrete space made of polynomials that are completely discontinuous across mesh elements

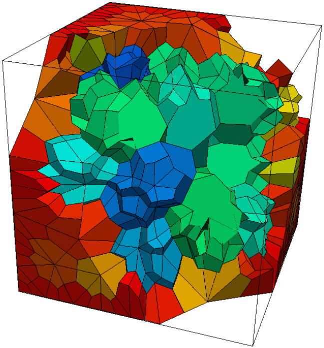
$$V_h \not\subseteq V$$



Features of DG Methods

- √ Wide range of PDE's treated within the same unified framework
- √ Weak approximation of boundary conditions
- √ Flexibility in mesh design
- √ Flexibility in polynomial degree distribution





- X Higher number of degrees of freedom
- X Larger algebraic linear systems to be solved (need of fast solvers)

Historical roots

- Introduced in the 70's for purely hyperbolic problems (Reed-Hill, Lesaint-Raviart)
- Extended in the mid 70's to second order elliptic PDEs (Douglas-Dupont) and to fourth order problems (Baker)
- Abandoned in 80's-90's due to a much larger number of degrees of freedom compared to their conforming cousins
- Great revival since the late 90's, application to a wide range of problems (linear/non-linear PDEs, time-dependent/stationary problems, spectrum approximation, multi-physics problems)

Essential Bibliography: Books

- B. Rivière. Discontinuous Galerkin Methods for Solving Elliptic and Parabolic Equations: Theory and Implementation, SIAM, 2008.
- J.S. Hesthaven, T. Warburton, Discontinuous Galerkin Methods. Algorithms, Analysis, and Applications, Springer, 2008.
- H. Fahs, High-order discontinuous Galerkin methods for the Maxwell equations, 2010, Editions Universitaires Europèennes,
- D. A. Di Pietro, A. Ern, Mathematical Aspects of Discontinuous Galerkin Methods, Springer, 2012

Essential Bibliography

- Elliptic/parabolic PDEs
  - Babuška & Zlámal (1973)
  - Douglas & Dupont (1976)
  - Baker (1977)
  - Wheeler (1978), Riviére & Wheeler (1999 $\longrightarrow$)
  - Arnold (1979, 1982)
  - Cockburn, Perugia & Schötzau (2000 $\longrightarrow$)
  - Arnold, Brezzi, Cockburn & Marini (SINUM, 2001/2002 $\longrightarrow$)
- Hyperbolic PDEs
  - Reed & Hill (1973, Los Alamos Technical report)
  - Lesaint & Raviart (1974, 1978)
  - Johnson, Nävert & Pitkäranta (1984), Johnson & Pitkäranta (1986)
  - Baumann (1997), Baumann & Oden (1997 $\longrightarrow$)
  - Cockburn & Shu (1989 $\longrightarrow$)
  - Houston & Süli (1999 $\longrightarrow$)

A first look at DG finite element methods

$$\text{Model problem} \qquad \longrightarrow \qquad \begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

- Take the equation $-\Delta u = f$, multiply it by a (elementwise smooth) test function v and integrate over an element $\mathcal{K} \in \mathcal{T}_h$

$$\int_{\mathcal{K}} -\Delta u v = \int_{\mathcal{K}} f v$$

- Integrate by parts and sum over all the elements $\mathcal{K} \in \mathcal{T}_h$

$$\sum_{\mathcal{K} \in \mathcal{T}_h} \int_{\mathcal{K}} \nabla u \cdot \nabla v - \sum_{\mathcal{K} \in \mathcal{T}_h} \int_{\partial\mathcal{K}} \nabla u \cdot \mathbf{n}_{\mathcal{K}} v = \int_{\Omega} f v$$

- To deal with the boxed term we need some further notation

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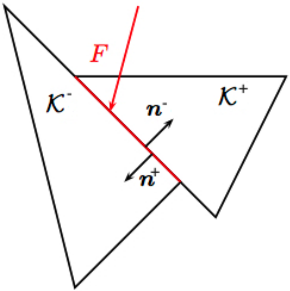
Trace Operators



- for any $F \in \mathcal{F}^I$(= set of interior faces) shared by $\mathcal{K}^{\pm}$

$$\{\!\{v\}\!\} = (v^+ + v^-)/2 \qquad [\![v]\!] = v^+\boldsymbol{n}^+ + v^-\boldsymbol{n}^-$$

$$\{\!\{\boldsymbol{\tau}\}\!\} = (\boldsymbol{\tau}^+ + \boldsymbol{\tau}^-)/2 \qquad [\![\boldsymbol{\tau}]\!] = \boldsymbol{\tau}^+ \cdot \boldsymbol{n}^+ + \boldsymbol{\tau}^- \cdot \boldsymbol{n}^-$$

- for any $F \in \mathcal{F}^B$(= set of boundary faces)

$$\{\!\{v\}\!\} = v \qquad [\![v]\!] = v\boldsymbol{n} \qquad \{\!\{\boldsymbol{\tau}\}\!\} = \boldsymbol{\tau} \qquad [\![\boldsymbol{\tau}]\!] = \boldsymbol{\tau} \cdot \boldsymbol{n}$$

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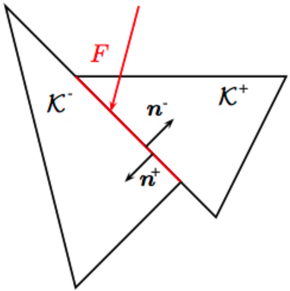

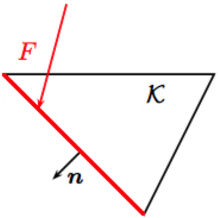

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Magic Formula (Arnold, 82)

$$\sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\partial\mathcal{K}} \boldsymbol{\tau}\cdot\mathbf{n}_{\mathcal{K}}\, v = \sum_{F\in\mathcal{F}}\int_F \{\!\{\boldsymbol{\tau}\}\!\}\cdot[\![v]\!] + \sum_{F\in\mathcal{F}^I}\int_F [\![\boldsymbol{\tau}]\!]\,\{\!\{v\}\!\}$$

Magic Formula (Arnold, 82)

$$\underbrace{\sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\partial\mathcal{K}}\boldsymbol{\tau}\cdot\mathbf{n}_{\mathcal{K}}v}_{(A)} = \underbrace{\sum_{F\in\mathcal{F}}\int_F \{\!\{\boldsymbol{\tau}\}\!\}\cdot[\![v]\!] + \sum_{F\in\mathcal{F}^I}\int_F [\![\boldsymbol{\tau}]\!]\{\!\{v\}\!\}}_{(B)}$$

Proof

Observe that

$$(A) = \sum_{F\in\mathcal{F}^I}\int_F(\boldsymbol{\tau}^+\cdot\mathbf{n}^+v^+ + \boldsymbol{\tau}^-\cdot\mathbf{n}^-v^-) + \sum_{F\in\mathcal{F}^B}\int_F\boldsymbol{\tau}\cdot\mathbf{n}v$$

$$(B) = \sum_{F\in\mathcal{F}^I}\int_F(\{\!\{\boldsymbol{\tau}\}\!\}\cdot[\![v]\!] + [\![\boldsymbol{\tau}]\!]\{\!\{v\}\!\}) + \sum_{F\in\mathcal{F}^B}\int_F\boldsymbol{\tau}\cdot\mathbf{n}v$$

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Magic Formula: proof (cont'd) (Arnold, 82)

Proof

Therefore, it is enough to show that on each internal face $F \in \mathcal{F}^I$

$$\int_F (\boldsymbol{\tau}^+ \cdot \mathbf{n}^+ v^+ + \boldsymbol{\tau}^- \cdot \mathbf{n}^- v^-) = \int_F (\{\!\{\boldsymbol{\tau}\}\!\} \cdot [\![v]\!] + [\![\boldsymbol{\tau}]\!] \{\!\{v\}\!\})$$

Magic Formula: proof (cont'd) (Arnold, 82)

Proof

Therefore, it is enough to show that on each internal face $F \in \mathcal{F}^I$

$$\int_F (\boldsymbol{\tau}^+ \cdot \mathbf{n}^+ v^+ + \boldsymbol{\tau}^- \cdot \mathbf{n}^- v^-) = \underbrace{\int_F (\{\!\{\boldsymbol{\tau}\}\!\} \cdot [\![v]\!] + [\![\boldsymbol{\tau}]\!] \{\!\{v\}\!\})}_{(C)}$$

Using the definition of the jump and average operators and that $\mathbf{n}^+ = -\mathbf{n}^-$ we have

$$\begin{aligned}(C) &= \frac{1}{2}\int_F (\boldsymbol{\tau}^+ + \boldsymbol{\tau}^-)(v^+ - v^-) \cdot \mathbf{n}^+ + (v^+ + v^-)(\boldsymbol{\tau}^+ - \boldsymbol{\tau}^-) \cdot \mathbf{n}^+ \\ &= \frac{1}{2}\int_F (2\boldsymbol{\tau}^+ v^+ + \boldsymbol{\tau}^- v^+ - \boldsymbol{\tau}^+ v^- - 2\boldsymbol{\tau}^- v^- + v^- \boldsymbol{\tau}^+ - v^+ \boldsymbol{\tau}^-) \cdot \mathbf{n}^+\end{aligned}$$

Magic Formula: proof (cont'd) (Arnold, 82)

Proof

Therefore, it is enough to show that on each internal face $F \in \mathcal{F}^I$

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$$\begin{aligned}(C) &= \frac{1}{2}\int_F (\boldsymbol{\tau}^+ + \boldsymbol{\tau}^-)(v^+ - v^-) \cdot \mathbf{n}^+ + (v^+ + v^-)(\boldsymbol{\tau}^+ - \boldsymbol{\tau}^-) \cdot \mathbf{n}^+ \\ &= \frac{1}{2}\int_F (2\boldsymbol{\tau}^+ v^+ \cancel{+\boldsymbol{\tau}^- v^+} \cancel{-\boldsymbol{\tau}^+ v^-} - 2\boldsymbol{\tau}^- v^- \cancel{+v^- \boldsymbol{\tau}^+} \cancel{-v^+ \boldsymbol{\tau}^-}) \cdot \mathbf{n}^+ \\ &= \int_F (\boldsymbol{\tau}^+ \cdot \mathbf{n}^+ v^+ + \boldsymbol{\tau}^- \cdot \mathbf{n}^- v^-)\end{aligned}$$

A first look at DG finite element methods (cont'd)

Magic formula

$$\sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\partial\mathcal{K}} \boldsymbol{\tau}\cdot\mathbf{n}_{\mathcal{K}} v = \sum_{F\in\mathcal{F}}\int_F \{\!\{\boldsymbol{\tau}\}\!\}\cdot[\![v]\!] + \sum_{F\in\mathcal{F}^I}\int_F [\![\boldsymbol{\tau}]\!]\{\!\{v\}\!\}$$

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A first look at DG finite element methods (cont'd)

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$$\sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\mathcal{K}}\nabla u\cdot\nabla v-\sum_{F\in\mathcal{F}}\int_F\{\!\{\nabla u\}\!\}\cdot[\![v]\!]-\sum_{F\in\mathcal{F}^I}\int_F[\![\nabla u]\!]\{\!\{v\}\!\}=\int_\Omega fv$$

A first look at DG finite element methods (cont'd)

Magic formula

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$$\sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\mathcal{K}} \nabla u\cdot\nabla v - \sum_{F\in\mathcal{F}}\int_F \{\!\{\nabla u\}\!\}\cdot[\![v]\!] + \sum_{F\in\mathcal{F}^I}\int_F [\![\nabla u]\!]\{\!\{v\}\!\} = \int_\Omega fv$$

Since $u\in H^2(\Omega)$, then $[\![\nabla u]\!]=0\ \forall\, F\in\mathcal{F}^I$.

A first look at DG finite element methods (cont'd)

- Use that $[\![u]\!] = 0 \; \forall F \in \mathcal{F}$ (since $u \in H_0^1(\Omega)$) to add a symmetry term

$$\sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{K}} \nabla u \cdot \nabla v - \sum_{F\in\mathcal{F}} \int_F \{\!\{\nabla u\}\!\} \cdot [\![v]\!] - \sum_{F\in\mathcal{F}} \int_F \{\!\{\nabla_h v\}\!\} \cdot [\![u]\!] = \int_\Omega fv$$

  where ∇_h is the elementwise gradient (v is only piecewise smooth).

- We also add a stabilization term that controls the jumps

$$\sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{K}} \nabla u \cdot \nabla v - \sum_{F\in\mathcal{F}} \int_F \{\!\{\nabla u\}\!\} \cdot [\![v]\!] - \sum_{F\in\mathcal{F}} \int_F [\![u]\!] \cdot \{\!\{\nabla_h v\}\!\} + \sum_{F\in\mathcal{F}} \int_F \gamma [\![u]\!] \cdot [\![v]\!] = \int_\Omega fv$$

  where γ is a stabilization function (that might depend on the discretization parameters) [Douglas-Dupont, Wheeler, Arnold], see later ......

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A first look at DG finite element methods (cont'd)

- For $p \geq 1$, define the DG discrete space

$$V_h^p = \{ v_h \in L^2(\Omega) : v_h|_T \in \mathbb{P}^p(T) \quad \forall T \in \mathcal{T}_h \} \boxed{\not\subseteq H_0^1(\Omega)}$$

- Discretize $u \rightsquigarrow u_h$, $v \rightsquigarrow v_h$

A first look at DG finite element methods (cont'd)

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- Discretize $u \rightsquigarrow u_h$, $v \rightsquigarrow v_h$

$$\text{Find } u_h \in V_h^p \quad \text{s.t.} \quad \mathcal{A}(u_h, v_h) = \int_\Omega f v_h \quad \forall\, v_h \in V_h^p$$

$$\mathcal{A}(w, v) = \sum_{\mathcal{K} \in \mathcal{T}_h} \int_{\mathcal{K}} \nabla w \cdot \nabla v - \sum_{F \in \mathcal{F}} \int_F \{\!\{\nabla_h w\}\!\} \cdot [\![v]\!] - \sum_{F \in \mathcal{F}} \int_F [\![w]\!] \cdot \{\!\{\nabla_h v\}\!\} + \sum_{F \in \mathcal{F}} \int_F \gamma\, [\![w]\!] \cdot [\![v]\!]$$

The class of Interior Penalty DG methods

$$\text{Find } u_h \in V_h^p \quad \text{s.t.} \quad \mathcal{A}(u_h, v_h) = \int_\Omega f v_h \quad \forall\, v_h \in V_h^p$$

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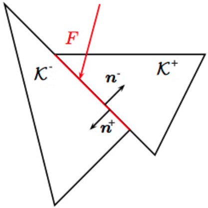
- $\theta = 1$: Symmetric Interior Penalty (SIP). [Wheeler, 78],[Arnold, 82]
- $\theta = -1$: Non-symmetric Interior Penalty (NIP). [Riviére, Wheeler & Girault, 99]
- $\theta = 0$: Incomplete Interior Penalty (IIP). [Dawson, Sun, Wheeler, 04]

The stabilization function γ

$$\sum_{F \in \mathcal{F}} \int_F \gamma [\![w]\!] \cdot [\![v]\!] \qquad \gamma = \alpha \frac{\mathtt{p}^2}{\mathtt{h}}$$



$$\mathtt{p} = \begin{cases} \max\{p_{\mathcal{K}^-}, p_{\mathcal{K}^-}\} & \text{if } F \in \mathcal{F}_h^I \\ p_{\mathcal{K}} & \text{if } F \in \mathcal{F}_h^B \end{cases} \qquad \mathtt{h} = \begin{cases} \min\{h_{\mathcal{K}^+}, h_{\mathcal{K}^-}\} & \text{if } F \in \mathcal{F}_h^I \\ h_{\mathcal{K}} & \text{if } F \in \mathcal{F}_h^B \end{cases}$$

Assumptions

$$p_{\mathcal{K}^+} \approx p_{\mathcal{K}^-} \qquad h_{\mathcal{K}^+} \approx h_{\mathcal{K}^-}$$

A little bit of theory: notation

- For an interger $s \geq 1$, define the broken Sobolev space

$$H^s(\mathcal{T}_h) = \left\{ v \in L^2(\Omega) \,:\, v|_{\mathcal{K}} \in H^s(\mathcal{K}) \ \forall \mathcal{K} \in \mathcal{T}_h \right\}$$

$$\|v\|^2_{H^s(\mathcal{T}_h)} = \sum_{\mathcal{K} \in \mathcal{T}_h} \|v\|^2_{H^s(\mathcal{K})}$$

- Define also

$$\|v\|^2_{L^2(\mathcal{F}_h)} = \sum_{F \in \mathcal{F}_h} \|v\|^2_{L^2(F)}$$

A little bit of theory: notation (cont'd)

- Define $\widetilde{V}_h^p = V_h^p + H^2(\mathcal{T}_h)$ and

$$\|v_h\|_{\mathrm{DG}}^2 = \|\nabla_h v_h\|_{L^2(\Omega)}^2 + \left\|\gamma^{1/2}\, [\![v_h]\!]\right\|_{L^2(\mathcal{F}_h)}^2 \quad \forall\, v_h \in V_h^p$$

$$|\!|\!| v |\!|\!|_{\mathrm{DG}}^2 = \|v\|_{\mathrm{DG}}^2 + \left\|\gamma^{-1/2}\, \{\!\{ \nabla_h v \}\!\}\right\|_{L^2(\mathcal{F}_h)}^2 \quad \forall\, v \in \widetilde{V}_h^p$$

It can be shown

$$\|v_h\|_{\mathrm{DG}} \underset{(trivial)}{\leq} |\!|\!| v_h |\!|\!|_{\mathrm{DG}} \underset{\text{(see later)}}{\lesssim} \|v_h\|_{\mathrm{DG}} \qquad \forall\, v_h \in V_h^p.$$

Key ingredients

1. Continuity on $\widetilde{V}_h^p$:

$$|\mathcal{A}(v,w)| \lesssim |||v|||_{\mathrm{DG}} |||w|||_{\mathrm{DG}} \qquad \forall\, v,w \in \widetilde{V}_h^p$$

2. Coercivity on V_h^p:

$$\mathcal{A}(v_h,v_h) \gtrsim \|v_h\|_{\mathrm{DG}}^2 \qquad \forall\, v_h \in V_h^p$$

3. Consisentcy

$$\mathcal{A}(u,v_h) = \int_\Omega f v_h \qquad \forall\, v_h \in V_h^p$$

4. Approximation. Let $\Pi_h^p u \in V_h^p$ a suitable approximation of u. Then

$$\left|\left|\left| u - \Pi_h^p u \right|\right|\right|_{\mathrm{DG}} \lesssim ??$$

Abstract error estimate

For the analysis, suppose, for simplicity, that

- the exact solution u is (at least) $u \in H^2(\Omega)$
- the mesh $\mathcal{T}_h$ is quasi-uniform (h=mesh-size)
- $p_{\mathcal{K}} = p$ for any $\mathcal{K} \in \mathcal{T}_h$

$$\|u - u_h\|_{\mathrm{DG}} \leq \left\|u - \Pi_h^p u\right\|_{\mathrm{DG}} + \left\|\Pi_h^p u - u_h\right\|_{\mathrm{DG}} \quad \text{(triangle inequality)}$$

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$$\begin{aligned}
\left\|\Pi_h^p u - u_h\right\|_{\mathrm{DG}}^2 &\lesssim \mathcal{A}(\Pi_h^p u - u_h, \Pi_h^p u - u_h) && \text{(Coercivity on } V_h^p\text{)} \\
&\lesssim \mathcal{A}(\Pi_h^p u - u, \Pi_h^p u - u_h) && \text{(Consistency)} \\
&\lesssim \left|\left|\left|\Pi_h^p u - u\right|\right|\right|_{\mathrm{DG}} \left|\left|\left|\Pi_h^p u - u_h\right|\right|\right|_{\mathrm{DG}} && \text{(Continuity on } \widetilde{V}_h^p\text{)} \\
&\lesssim \left|\left|\left|\Pi_h^p u - u\right|\right|\right|_{\mathrm{DG}} \left\|\Pi_h^p u - u_h\right\|_{\mathrm{DG}} && \text{(Norms equivalence on } V_h^p\text{)}
\end{aligned}$$

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- the exact solution u is (at least) $u \in H^2(\Omega)$
- the mesh $\mathcal{T}_h$ is quasi-uniform (h=mesh-size)
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$$\begin{aligned}
\left\|\Pi_h^p u - u_h\right\|_{\mathrm{DG}}^2 &\lesssim \mathcal{A}(\Pi_h^p u - u_h, \Pi_h^p u - u_h) && \text{(Coercivity on } V_h^p\text{)} \\
&\lesssim \mathcal{A}(\Pi_h^p u - u, \Pi_h^p u - u_h) && \text{(Consistency)} \\
&\lesssim \left|\!\left|\!\left|\Pi_h^p u - u\right|\!\right|\!\right|_{\mathrm{DG}} \left|\!\left|\!\left|\Pi_h^p u - u_h\right|\!\right|\!\right|_{\mathrm{DG}} && \text{(Continuity on } \widetilde{V}_h^p\text{)} \\
&\lesssim \left|\!\left|\!\left|\Pi_h^p u - u\right|\!\right|\!\right|_{\mathrm{DG}} \left\|\Pi_h^p u - u_h\right\|_{\mathrm{DG}} && \text{(Norms equivalence on } V_h^p\text{)}
\end{aligned}$$

Abstract error estimate

For the analysis, suppose, for simplicity, that

- the exact solution u is (at least) $u \in H^2(\Omega)$
- the mesh $\mathcal{T}_h$ is quasi-uniform (h=mesh-size)
- $p_{\mathcal{K}} = p$ for any $\mathcal{K} \in \mathcal{T}_h$

$$\|u - u_h\|_{\mathrm{DG}} \leq \left\|u - \Pi_h^p u\right\|_{\mathrm{DG}} + \left\|\Pi_h^p u - u_h\right\|_{\mathrm{DG}} \quad \text{(triangle inequality)}$$

$$\begin{aligned}
\left\|\Pi_h^p u - u_h\right\|_{\mathrm{DG}}^2 &\lesssim \mathcal{A}(\Pi_h^p u - u_h, \Pi_h^p u - u_h) && \text{(Coercivity on } V_h^p\text{)} \\
&\lesssim \mathcal{A}(\Pi_h^p u - u, \Pi_h^p u - u_h) && \text{(Consistency)} \\
&\lesssim \left|\!\left|\!\left|\Pi_h^p u - u\right|\!\right|\!\right|_{\mathrm{DG}} \left|\!\left|\!\left|\Pi_h^p u - u_h\right|\!\right|\!\right|_{\mathrm{DG}} && \text{(Continuity on } \widetilde{V}_h^p\text{)} \\
&\lesssim \left|\!\left|\!\left|\Pi_h^p u - u\right|\!\right|\!\right|_{\mathrm{DG}} \left\|\Pi_h^p u - u_h\right\|_{\mathrm{DG}} && \text{(Norms equivalence on } V_h^p\text{)}
\end{aligned}$$

$$\|u - u_h\|_{\mathrm{DG}} \lesssim \left|\!\left|\!\left|u - \Pi_h^p u\right|\!\right|\!\right|_{\mathrm{DG}}$$

Preliminary (key) estimate

Let $v, w \in \widetilde{V}_h^p$, we want to estimate a term of the form

$$\left| \sum_{F \in \mathcal{F}} \int_F \{\!\{\nabla_h w\}\!\} \cdot [\![v]\!] \right|$$

Via Cauchy-Schwarz inequality $(\sum a_i b_i \leq (\sum a_i^2)^{1/2} (\sum b_i^2)^{1/2})$ we get

$$\begin{aligned}
\left| \sum_{F \in \mathcal{F}} \int_F \{\!\{\nabla_h w\}\!\} \cdot [\![v]\!] \right| &= \left| \sum_{F \in \mathcal{F}} \int_F \gamma^{-1/2} \{\!\{\nabla_h w\}\!\} \cdot \gamma^{1/2} [\![v]\!] \right| \\
&\leq \left(\sum_{F \in \mathcal{F}} \left\| \gamma^{-1/2} \; \{\!\{\nabla_h w\}\!\} \right\|_{L^2(F)}^2 \right)^{1/2} \left(\sum_{F \in \mathcal{F}} \left\| \gamma^{1/2} \; [\![v]\!] \right\|_{L^2(F)}^2 \right)^{1/2} \\
&= \left\| \gamma^{-1/2} \; \{\!\{\nabla_h w\}\!\} \right\|_{L^2(\mathcal{F}_h)} \left\| \gamma^{1/2} [\![v]\!] \right\|_{L^2(\mathcal{F}_h)}
\end{aligned}$$

Preliminary (key) estimate (cont'd)

$$\left|\sum_{F\in\mathcal{F}}\int_F \{\!\{\nabla_h w\}\!\}\cdot[\![v]\!]\right| \leq \left\|\gamma^{-1/2}\ \{\!\{\nabla_h w\}\!\}\right\|_{L^2(\mathcal{F}_h)}\left\|\gamma^{1/2}\,[\![v]\!]\right\|_{L^2(\mathcal{F}_h)} \quad \forall\, w,v\in\widetilde{V}_h^p$$

If w is piecewise polynomial, the term $\left\|\gamma^{-1/2}\ \{\!\{\nabla_h w\}\!\}\right\|_{L^2(\mathcal{F}_h)}$ can be further manipulated using the following trace/inverse estimate

$$\|\nabla\eta\|_{L^2(F)} \lesssim \frac{p}{h^{1/2}}\|\nabla\eta\|_{L^2(\mathcal{K})} \qquad \forall\, F\subset\partial\mathcal{K} \qquad \forall\,\eta\in\mathbb{P}^p(\mathcal{K})$$

Indeed, if $w\in V_h^p$ we have

$$\begin{aligned}\left\|\gamma^{-1/2}\ \{\!\{\nabla_h w\}\!\}\right\|^2_{L^2(\mathcal{F}_h)} &\lesssim \frac{h}{\alpha p^2}\sum_{F\in\mathcal{F}_h}\|\{\!\{\nabla_h w\}\!\}\|^2_{L^2(F)} \\ &\lesssim \frac{h}{\alpha p^2}\frac{p^2}{h}\sum_{\mathcal{K}\in\mathcal{T}_h}\|\nabla_h w\|^2_{L^2(\mathcal{K}^+\cup\mathcal{K}^-)} \lesssim \frac{1}{\alpha}\|\nabla_h w\|^2_{L^2(\Omega)}\end{aligned}$$

Preliminary (key) estimate (cont'd)

In summary

$$\left|\sum_{F\in\mathcal{F}}\int_F \{\!\{\nabla_h w\}\!\}\cdot[\![v]\!]\right| \leq \left\|\gamma^{-1/2}\ \{\!\{\nabla_h w\}\!\}\right\|_{L^2(\mathcal{F}_h)}\left\|\gamma^{1/2}\,[\![v]\!]\right\|_{L^2(\mathcal{F}_h)} \qquad \forall\, w, v \in \widetilde{V}_h^p$$

$$\left|\sum_{F\in\mathcal{F}}\int_F \{\!\{\nabla_h w\}\!\}\cdot[\![v]\!]\right| \lesssim \frac{1}{\sqrt{\alpha}}\|\nabla_h w\|_{L^2(\Omega)}\left\|\gamma^{1/2}\,[\![v]\!]\right\|_{L^2(\mathcal{F}_h)} \qquad \forall\, w, v \in V_h^p$$

Key ingredients

1. Continuity on $\widetilde{V}_h^p$:

$$|\mathcal{A}(v,w)| \lesssim |||v|||_{\mathrm{DG}} |||w|||_{\mathrm{DG}} \qquad \forall\, v,w \in \widetilde{V}_h^p$$

2. Coercivity on V_h^p:

$$\mathcal{A}(v_h,v_h) \gtrsim \|v_h\|_{\mathrm{DG}}^2 \qquad \forall\, v_h \in V_h^p$$

3. Consistency

$$\mathcal{A}(u,v_h) = \int_\Omega f v_h \qquad \forall\, v_h \in V_h^p$$

4. Approximation. Let $\Pi_h^p u \in V_h^p$ a suitable approximation of u. Then

$$\left|\left|\left| u - \Pi_h^p u \right|\right|\right|_{\mathrm{DG}} \lesssim ??$$

Continuity on $\widetilde{V}_h^p$

$$|\mathcal{A}(v,w)| \lesssim |||v|||_{\mathrm{DG}} |||w|||_{\mathrm{DG}} \qquad \forall\, v,w \in \widetilde{V}_h^p$$

Continuity on $\widetilde{V}_h^p$

Let $v, w \in \widetilde{V}_h^p$, then

$$|\mathcal{A}(v,w)| \leq \underbrace{\left|\sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\mathcal{K}} \nabla w \cdot \nabla v\right|}_{(I)} + \underbrace{\left|\sum_{F\in\mathcal{F}}\int_F \{\!\{\nabla_h w\}\!\} \cdot [\![v]\!]\right|}_{(II)}$$

$$+ \underbrace{\left|\sum_{F\in\mathcal{F}}\int_F [\![w]\!] \cdot \{\!\{\nabla_h v\}\!\}\right|}_{(III)} + \underbrace{\left|\sum_{F\in\mathcal{F}}\int_F \gamma\, [\![w]\!] \cdot [\![v]\!]\right|}_{(IV)}$$

Continuity on $\widetilde{V}_h^p$ (cont'd)

Via Cauchy-Schwarz inequality ($\sum a_i b_i \leq (\sum a_i^2)^{1/2}(\sum b_i^2)^{1/2}$) we get

$$
\begin{aligned}
(I) &\leq \left(\sum_{\mathcal{K}\in\mathcal{T}_h} \|\nabla_h w\|^2_{L^2(\mathcal{K})} \right)^{1/2} \left(\sum_{\mathcal{K}\in\mathcal{T}_h} \|\nabla_h v\|^2_{L^2(\mathcal{K})} \right)^{1/2} \\
&= \|\nabla_h w\|_{L^2(\Omega)} \|\nabla_h v\|_{L^2(\Omega)} \leq |||w|||_{\mathrm{DG}} |||v|||_{\mathrm{DG}}
\end{aligned}
$$

Analogously

$$
\begin{aligned}
(IV) &\leq \left(\sum_{F\in\mathcal{F}_h} \left\|\gamma^{1/2} [\![w]\!]\right\|^2_{L^2(F)} \right)^{1/2} \left(\sum_{F\in\mathcal{F}_h} \left\|\gamma^{1/2} [\![w]\!]\right\|^2_{L^2(F)} \right)^{1/2} \\
&= \left\|\gamma^{1/2} [\![w]\!]\right\|_{L^2(\mathcal{F}_h)} \left\|\gamma^{1/2} [\![v]\!]\right\|_{L^2(\mathcal{F}_h)} \leq |||w|||_{\mathrm{DG}} |||v|||_{\mathrm{DG}}
\end{aligned}
$$

Continuity on $\widetilde{V}_h^p$ (cont'd)

$$(II) \leq \left(\sum_{F \in \mathcal{F}_h} \left\| \gamma^{-1/2} \{\!\{ \nabla_h w \}\!\} \right\|^2_{L^2(F)} \right)^{1/2} \left(\sum_{F \in \mathcal{F}_h} \left\| \gamma^{1/2} [\![v]\!] \right\|^2_{L^2(F)} \right)^{1/2}$$

$$= \left\| \gamma^{-1/2} \{\!\{ \nabla_h w \}\!\} \right\|_{L^2(\mathcal{F}_h)} \left\| \gamma^{1/2} [\![v]\!] \right\|_{L^2(\mathcal{F}_h)} \leq |\!|\!| w |\!|\!|_{\mathrm{DG}} |\!|\!| v |\!|\!|_{\mathrm{DG}}$$

$$(II) \leq \left(\sum_{F \in \mathcal{F}_h} \left\| \gamma^{-1/2} \{\!\{ \nabla_h v \}\!\} \right\|^2_{L^2(F)} \right)^{1/2} \left(\sum_{F \in \mathcal{F}_h} \left\| \gamma^{1/2} [\![w]\!] \right\|^2_{L^2(F)} \right)^{1/2}$$

$$= \left\| \gamma^{-1/2} \{\!\{ \nabla_h v \}\!\} \right\|_{L^2(\mathcal{F}_h)} \left\| \gamma^{1/2} [\![w]\!] \right\|_{L^2(\mathcal{F}_h)} \leq |\!|\!| v |\!|\!|_{\mathrm{DG}} |\!|\!| w |\!|\!|_{\mathrm{DG}}$$

Coercivity on V_h^p

$$\mathcal{A}(v_h, v_h) \gtrsim \|v_h\|_{\mathrm{DG}}^2 \qquad \forall\, v_h \in V_h^p$$

Coercivity on V_h^p

Recalling the definition of $\mathcal{A}(\cdot,\cdot)$, we have

$$\mathcal{A}(v_h, v_h) = \|\nabla_h v_h\|^2_{L^2(\Omega)} - (1+\theta) \sum_{F\in\mathcal{F}} \int_F \{\!\{\nabla_h v_h\}\!\} \cdot [\![v_h]\!] + \sum_{F\in\mathcal{F}_h} \left\| \gamma^{1/2} \, [\![v_h]\!] \right\|^2_{0,F}$$

- If $\theta = -1$ (i.e. NIP method), the result is trivial as

$$\mathcal{A}(v_h, v_h) = \|\nabla_h v_h\|^2_{L^2(\Omega)} + \left\| \gamma^{1/2} \, [\![v_h]\!] \right\|^2_{0,F} = \|v_h\|^2_{\mathrm{DG}}$$

- If $\theta = 0, 1$ (i.e. SIP/IIP methods), we have to estimate the term

$$(I) = \sum_{F\in\mathcal{F}} \int_F \{\!\{\nabla_h v_h\}\!\} \cdot [\![v_h]\!] \le \left| \sum_{F\in\mathcal{F}} \int_F \{\!\{\nabla_h v_h\}\!\} \cdot [\![v_h]\!] \right|$$
$$= \left\| \gamma^{-1/2} \, \{\!\{\nabla_h v_h\}\!\} \right\|_{L^2(\mathcal{F}_h)} \left\| \gamma^{1/2} \, [\![v_h]\!] \right\|_{L^2(\mathcal{F}_h)}$$

Coercivity on V_h^p (cont'd)

From the arithmetic-geometric inequality we have, for any $\varepsilon > 0$,

$$(I) \leq \frac{\varepsilon}{2}\left\|\gamma^{-1/2}\ \{\!\{\nabla_h v_h\}\!\}\right\|^2_{L^2(\mathcal{F}_h)} + \frac{1}{2\varepsilon}\left\|\gamma^{1/2}\ [\![v_h]\!]\right\|^2_{L^2(\mathcal{F}_h)}$$

From which it follows

$$\begin{aligned}
\mathcal{A}(v_h, v_h) &= \|\nabla_h v_h\|^2_{L^2(\Omega)} - (1+\theta)\sum_{F\in\mathcal{F}}\int_F \{\!\{\nabla_h v_h\}\!\}\cdot[\![v_h]\!] + \left\|\gamma^{1/2}\ [\![v_h]\!]\right\|^2_{L^2(\mathcal{F}_h)} \\
&\geq \|\nabla_h v_h\|^2_{L^2(\Omega)} - (1+\theta)\left|\sum_{F\in\mathcal{F}}\int_F \{\!\{\nabla_h v_h\}\!\}\cdot[\![v_h]\!]\right| + \left\|\gamma^{1/2}\ [\![v_h]\!]\right\|^2_{L^2(\mathcal{F}_h)} \\
&\geq \|\nabla_h v_h\|^2_{L^2(\Omega)} - (1+\theta)\frac{\varepsilon}{2}\left\|\gamma^{-1/2}\ \{\!\{\nabla_h v_h\}\!\}\right\|^2_{L^2(\mathcal{F}_h)} + \left(1-\frac{1+\theta}{2\varepsilon}\right)\left\|\gamma^{1/2}\ [\![v_h]\!]\right\|^2_{L^2(\mathcal{F}_h)}
\end{aligned}$$

Coercivity on V_h^p (cont'd)

From the arithmetic-geometric inequality we have, for any $\varepsilon > 0$,

$$(I) \leq \frac{\varepsilon}{2} \left\| \gamma^{-1/2} \, \{\!\{\nabla_h v_h\}\!\} \right\|^2_{L^2(\mathcal{F}_h)} + \frac{1}{2\varepsilon} \left\| \gamma^{1/2} \, [\![v_h]\!] \right\|^2_{L^2(\mathcal{F}_h)}$$

From which it follows

$$\begin{aligned}
\mathcal{A}(v_h, v_h) &= \|\nabla_h v_h\|^2_{L^2(\Omega)} - (1+\theta) \sum_{F \in \mathcal{F}} \int_F \{\!\{\nabla_h v_h\}\!\} \cdot [\![v_h]\!] + \left\| \gamma^{1/2} \, [\![v_h]\!] \right\|^2_{L^2(\mathcal{F}_h)} \\
&\geq \|\nabla_h v_h\|^2_{L^2(\Omega)} - (1+\theta) \left| \sum_{F \in \mathcal{F}} \int_F \{\!\{\nabla_h v_h\}\!\} \cdot [\![v_h]\!] \right| + \left\| \gamma^{1/2} \, [\![v_h]\!] \right\|^2_{L^2(\mathcal{F}_h)} \\
&\geq \|\nabla_h v_h\|^2_{L^2(\Omega)} - (1+\theta) \frac{\varepsilon}{2} \boxed{\left\| \gamma^{-1/2} \, \{\!\{\nabla_h v_h\}\!\} \right\|^2_{L^2(\mathcal{F}_h)}} + \left(1 - \frac{1+\theta}{2\varepsilon}\right) \left\| \gamma^{1/2} \, [\![v_h]\!] \right\|^2_{L^2(\mathcal{F}_h)}
\end{aligned}$$

We now bound the term in the box

Coercivity on V_h^p (cont'd)

Trace/inverse estimate

$$\|\nabla \eta\|_{L^2(F)} \lesssim \frac{p}{h^{1/2}} \|\nabla \eta\|_{L^2(\mathcal{K})} \qquad \forall\, F \subset \partial \mathcal{K} \qquad \forall\, \eta \in \mathbb{P}^p(\mathcal{K})$$

Using the above trace-inverse estimate

$$\begin{aligned}
\left\| \gamma^{-1/2} \; \{\!\{ \nabla_h v_h \}\!\} \right\|^2_{L^2(\mathcal{F}_h)} &\lesssim \frac{h}{p^2 \alpha} \sum_{F \in \mathcal{F}_h} \| \{\!\{ \nabla_h v_h \}\!\} \|^2_{L^2(F)} \\
&\lesssim \alpha \frac{h}{\alpha p^2} \frac{p^2}{h} \sum_{\mathcal{K} \in \mathcal{T}_h} \| \nabla_h v_h \|_{L^2(\mathcal{K}^+ \cup \mathcal{K}^-)} \\
&\leq \frac{1}{\alpha} C_{\mathrm{I}} \| \nabla_h v_h \|_{L^2(\Omega)}
\end{aligned}$$

Coercivity on V_h^p (cont'd)

Therefore we get

$$\mathcal{A}(v_h, v_h) \geq \left(1 - \frac{C_{\mathrm{I}}(1+\theta)\varepsilon}{2\alpha}\right) \|\nabla_h v_h\|^2_{L^2(\Omega)} + \left(1 - \frac{1+\theta}{2\varepsilon}\right) \left\|\gamma^{1/2} [\![v_h]\!]\right\|^2_{L^2(\mathcal{F}_h)}$$

Coercivity on V_h^p (cont'd)

Therefore we get

$$\mathcal{A}(v_h, v_h) \geq \left(1 - \frac{C_{\mathrm{I}}(1+\theta)\varepsilon}{2\alpha}\right) \|\nabla_h v_h\|^2_{L^2(\Omega)} + \left(1 - \frac{1+\theta}{2\varepsilon}\right) \left\|\gamma^{1/2} \, [\![v_h]\!]\right\|^2_{L^2(\mathcal{F}_h)}$$

Now choose (for example) $\varepsilon = 1 + \theta$ to get

$$\mathcal{A}(v_h, v_h) \geq \left(1 - \frac{C_{\mathrm{I}}(1+\theta)^2}{2\alpha}\right) \|\nabla_h v_h\|^2_{L^2(\Omega)} + \frac{1}{2}\left\|\gamma^{1/2} \, [\![v_h]\!]\right\|^2_{L^2(\mathcal{F}_h)}$$

Coercivity on V_h^p (cont'd)

Therefore we get

$$\mathcal{A}(v_h, v_h) \geq \left(1 - \frac{C_{\mathrm{I}}(1+\theta)\varepsilon}{2\alpha}\right) \|\nabla_h v_h\|^2_{L^2(\Omega)} + \left(1 - \frac{1+\theta}{2\varepsilon}\right) \left\|\gamma^{1/2} [\![v_h]\!]\right\|^2_{L^2(\mathcal{F}_h)}$$

Now choose (for example) $\varepsilon = 1 + \theta$ to get

$$\mathcal{A}(v_h, v_h) \geq \left(1 - \frac{C_{\mathrm{I}}(1+\theta)^2}{2\alpha}\right) \|\nabla_h v_h\|^2_{L^2(\Omega)} + \frac{1}{2}\left\|\gamma^{1/2} [\![v_h]\!]\right\|^2_{L^2(\mathcal{F}_h)}$$

Therefore we obtain

$$\mathcal{A}(v_h, v_h) \geq \frac{1}{2}\|\nabla_h v_h\|^2_{L^2(\Omega)} + \frac{1}{2}\left\|\gamma^{1/2} [\![v_h]\!]\right\|^2_{L^2(\mathcal{F}_h)} = \frac{1}{2}\|v_h\|^2_{\mathrm{DG}}$$

provided

$$1 - \frac{C_{\mathrm{I}}(1+\theta)^2}{2\alpha} \geq \frac{1}{2} \qquad \Leftrightarrow \qquad \alpha \geq C_{\mathrm{I}}(1+\theta)^2 \equiv \alpha^*$$

Consistency

$$\mathcal{A}(u, v_h) = \int_{\Omega} f v_h \qquad \forall\, v_h \in V_h^p$$

Consistency

Recalling the Magic Formula, we have the following

Integration by parts formula

$$-\sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\mathcal{K}} \nabla\cdot\boldsymbol{\tau}\, v = \sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\mathcal{K}} \boldsymbol{\tau}\cdot\nabla v - \sum_{F\in\mathcal{F}}\int_F \{\!\{\boldsymbol{\tau}\}\!\}\cdot[\![v]\!] - \sum_{F\in\mathcal{F}^I}\int_F [\![\boldsymbol{\tau}]\!]\,\{\!\{v\}\!\}$$

for all $\boldsymbol{\tau}\in[H^1(\mathcal{T}_h)]^d$, $v\in H^1(\mathcal{T}_h)$

Now, let u solves the boundary value problem

$$-\Delta u = f \quad \text{in } \Omega \quad u = 0 \quad \text{on } \partial\Omega$$

Consistency (cont'd)

From the integration by parts formula we get

$$\sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\mathcal{K}}\nabla u\cdot\nabla v = -\sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\mathcal{K}}\Delta u v+\sum_{F\in\mathcal{F}}\int_F \{\!\{\nabla u\}\!\}\cdot[\![v]\!]+\sum_{F\in\mathcal{F}^I}\int_F [\![\nabla u]\!]\,\{\!\{v\}\!\}$$

for any $v\in H^1(\mathcal{T}_h)$. Since $[\![\nabla u]\!]=0$ on each interior face and $-\Delta u = f$, we obtain

$$\sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\mathcal{K}}\nabla u\cdot\nabla v - \sum_{F\in\mathcal{F}}\int_F \{\!\{\nabla u\}\!\}\cdot[\![v]\!] = \int_\Omega fv \qquad \forall\, v\in H^1(\mathcal{T}_h).$$

Consistency (cont'd)

Since $[\![u]\!] = 0$ on each interior face and $[\![u]\!] = u = 0$ on each boundary face, we can also write

$$\sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\mathcal{K}} \nabla u \cdot \nabla v - \sum_{F\in\mathcal{F}}\int_F \{\!\{\nabla u\}\!\} \cdot [\![v]\!] - \sum_{F\in\mathcal{F}}\int_F [\![u]\!] \cdot \{\!\{\nabla_h v\}\!\} + \sum_{F\in\mathcal{F}}\int_F \gamma\, [\![u]\!] \cdot [\![v]\!] = \int_\Omega fv \qquad \forall\, v \in H^1(\mathcal{T}_h).$$

That is

$$\mathcal{A}(u, v) = \int_\Omega fv \qquad \forall\, v \in H^1(\mathcal{T}_h).$$

Consistency (cont'd)

$$\mathcal{A}(u, v_h) = \int_\Omega f v_h \qquad \forall\, v \in V_h^p.$$

By linearity we than obtain

Galerkin orthogonality

$$\mathcal{A}(u - u_h, v_h) = 0 \qquad \forall\, v \in V_h^p.$$

Approximation bound for $|||u - \Pi_h^p u|||_{\mathrm{DG}}$

Approximation bound for $|||u - \Pi_h^p u|||_{\mathrm{DG}}$

Set $e_\pi = u - \Pi_h^p u$ and recall that

$$|||e_\pi|||_{\mathrm{DG}}^2 = \underbrace{\|\nabla_h e_\pi\|_{L^2(\Omega)}^2}_{(I)} + \underbrace{\sum_{F\in\mathcal{F}_h} \left\|\gamma^{1/2}\ [\![e_\pi]\!]\right\|_{0,F}^2}_{(II)} + \underbrace{\sum_{F\in\mathcal{F}_h} \left\|\gamma^{-1/2}\ \{\!\{\nabla_h e_\pi\}\!\}\right\|_{0,F}^2}_{(III)}$$

We make use of the following $hp-$approximation result

An $hp-$approximation result [Babuska, Suri, 1987]

Let $\mathcal{K} \in \mathcal{T}_h$ and let $v \in H^s(\mathcal{K})$, $s \geq 1$. Then, there exists a sequence of operators $\Pi_h^p v : H^s(\mathcal{K}) \longrightarrow \mathbb{P}^p(\mathcal{K})$, $p = 1, 2, \ldots$ such that

$$\left\| v - \Pi_h^p v \right\|_{H^q(\mathcal{K})} \lesssim \frac{h^{\min(p+1,s)-q}}{p^{s-q}} \|v\|_{H^s(\mathcal{K})} \quad 0 \leq q \leq s$$

The hidden constant is independent of v, h and p, but depends on the shape-regularity of $\mathcal{K}$ an on s.

For the proof, see [Babuska, Suri, 1987], for $d = 2$. The case $d = 3$ is analogous.

Approximation bound for $\|\!|u - \Pi_h^p u|\!\|_{\mathrm{DG}}$ (cont'd)

Then, for the term (I) we get

$$(I) = \left\|\nabla_h(u - \Pi_h^p u)\right\|_{L^2(\Omega)}^2 \lesssim \frac{h^{2\min(p+1,s)-2}}{p^{2s-2}}\|v\|_{H^s(\mathcal{T}_h)}^2$$

To estimate (II), we make use of the multiplicative trace inequality

Multiplicative trace inequality

$$\|\eta\|_{L^2(\partial\mathcal{K})}^2 \lesssim \|\eta\|_{L^2(\mathcal{K})}\|\nabla\eta\|_{L^2(\mathcal{K})} + h^{-1}\|\eta\|_{L^2(\mathcal{K})}^2 \quad \forall\,\eta \in H^1(\mathcal{K})$$

where the hidden constant depends on the shape regularity of $\mathcal{K}$ and on the dimension d.

$$
\begin{aligned}
(II) &= \sum_{F \in \mathcal{F}_h} \left\| \gamma^{1/2} \, [\![e_\pi]\!] \right\|_{0,F}^2 \lesssim \frac{p^2}{h} \sum_{\mathcal{K} \in \mathcal{T}_h} \|e_\pi\|_{L^2(\partial\mathcal{K})}^2 \\
&\lesssim \frac{p^2}{h} \sum_{\mathcal{K} \in \mathcal{T}_h} \|e_\pi\|_{L^2(\mathcal{K})} \|\nabla e_\pi\|_{L^2(\mathcal{K})} + h^{-1} \|e_\pi\|_{L^2(\mathcal{K})}^2 \\
&\lesssim \frac{p^2}{h} \frac{h^{2\min(p+1,s)-1}}{p^{2s-1}} \|u\|_{H^s(\mathcal{T}_h)}^2
\end{aligned}
$$

Analogously, we have

$$
\begin{aligned}
(III) &= \sum_{F \in \mathcal{F}_h} \left\| \gamma^{-1/2} \, \{\!\{ \nabla_h e_\pi \}\!\} \right\|_{0,F}^2 \lesssim \frac{p^2}{h} \sum_{\mathcal{K} \in \mathcal{T}_h} \|\nabla e_\pi\|_{L^2(\partial\mathcal{K})}^2 \\
&\lesssim \frac{p^2}{h} \frac{h^{2\min(p+1,s)-3}}{p^{2s-3}} \|u\|_{H^s(\mathcal{T}_h)}^2
\end{aligned}
$$

Approximation bound for $\|u - \Pi_h^p u\|_{\mathrm{DG}}$ (cont'd)

If the exact solution $u \in H^s(\mathcal{T}_h)$, $s \geq 2$, then

$$\left|\!\left|\!\left| u - \Pi_h^p u \right|\!\right|\!\right|_{\mathrm{DG}} \lesssim \frac{h^{\min(p+1,s)-1}}{p^{s-1/2}} \|u\|_{H^s(\mathcal{T}_h)}$$

In particular, if $p \geq s - 1$, the estimate becomes

$$\left|\!\left|\!\left| u - \Pi_h^p u \right|\!\right|\!\right|_{\mathrm{DG}} \lesssim \left(\frac{h}{p}\right)^{s-1} p^{1/2} \|u\|_{H^s(\mathcal{T}_h)}.$$

Abstract error estimate

If $u \in H^s(\mathcal{T}_h)$, $s \geq 2$, then

$$\|u - u_h\|_{\mathrm{DG}} \lesssim \frac{h^{\min(p+1,s)-1}}{p^{s-1/2}} \|u\|_{H^s(\mathcal{T}_h)}$$

For the SIP and IIP methods the above estimate holds provided that the penalty constant α is chosen sufficiently large.

- The bound is optimal in h and suboptimal in p by a factor $p^{1/2}$. See, for example, [Houston, Schwab, Suli, 2001], [Riviere, Wheeler, Girault, 1999], [Perugia, Schotzau, 2001].
- Optimal error estimates with respect to p can be shown using the projector of [Georgoulis & Suli, 2005] provided the solution belongs to a suitable augmented space, or whenever a continuous interpolant can be built; cf. [Stamm & Wihler, 2010].

L^2-norm error estimates by duality argument

An estimate for the L^2-error can be obtained by using a duality argument.

Elliptic regularity

Assume that Ω is such that the following ellitpic regularity result holds: for any $g \in L^2(\Omega)$, the solution z of the problem

$$-\Delta z = g \quad \text{in } \Omega \qquad\qquad z = 0 \quad \text{on } \partial\Omega$$

satisfies $z \in H^2(\Omega)$ and

$$\|z\|_{H^2(\Omega)} \lesssim \|g\|_{L^2(\Omega)}$$

L^2–norm error estimates by duality argument (cont'd)

If the exact solution $u \in H^s(\mathcal{T}_h)$, $s \geq 2$ and if u_h is the solution obtained with the SIP method ($\theta = 1$), it holds

$$\|u - u_h\|_{L^2(\Omega)} \lesssim \frac{h^{\min(p+1,s)}}{p^{s+1/2}} \|u\|_{H^s(\mathcal{T}_h)}$$

The essential ingredient in the duality argument for proving L^2 estimates is the following adjoint consistency property

$$\mathcal{A}(v_h, z) = \int_\Omega f v_h \quad \forall\, v_h \in V_h^p$$

L^2–norm error estimates by duality argument (cont'd)

For NIP and IIP formulations it only holds

$$\|u - u_h\|_{L^2(\Omega)} \lesssim \frac{h^{\min(p+1,s)-1}}{p^{s-1/2}} \|u\|_{H^s(\mathcal{T}_h)}$$

since the corresponding bilinear form is non-symmetric and does not satisfy the adjoint consistency property