Abstract

We prove the existence of a fundamental solution for a class of Hörmander heat-type operators. For this fundamental solution and its derivatives we obtain sharp Gaussian bounds that allow to prove an invariant Harnack inequality.

Résumé

Noyaux de la chaleur pour des opérateurs de Hörmander qui ne sont pas sous forme de divergence. Nous démontrons l’existence d’une solution fondamentale pour une classe d’opérateurs de Hörmander de type chaleur. Pour cette solution fondamentale et ses dérivées nous obtenons des bornes Gaussiennes optimales qui nous permettent de démontrer une inégalité de Harnack invariante.

1. Introduction and main results

In geometric theory of several complex variables, fully nonlinear second order equations appear, whose linearizations are nonvariational operators of Hörmander type. (See, for instance, [10], and references therein.) These kinds of operators, also arising in many other theoretical and applied settings, have the following form, in the stationary or evolutionary case:

\[ L_A = \sum_{i,j=1}^{q} a_{ij}(x) X_i X_j, \]

\[ H_A = \partial_t - \sum_{i,j=1}^{q} a_{ij}(t,x) X_i X_j \]

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where $X_1, X_2, \ldots, X_q$ is a system of real smooth vector fields defined in some bounded domain $\Omega \subset \mathbb{R}^n$ and satisfying Hörmander’s rank condition at any point. The matrix $A = \{a_{ij}(t, x)\}_{i,j=1}^q$ is real symmetric and uniformly positive definite, that is:

$$\lambda^{-1} |\xi|^2 \leq \sum_{i,j=1}^q a_{ij}(t, x) \xi_i \xi_j \leq \lambda |\xi|^2$$

for some $\lambda > 0$, every $\xi \in \mathbb{R}^q$, $(t, x) \in \mathbb{R} \times \Omega$. On the coefficients $a_{ij}$ we will make a natural assumption of Hölder continuity, expressed in terms of the distance induced by the vector fields. More precisely, if $d(x, y)$ denotes the Carnot–Carathéodory metric generated in $\mathbb{R}^n$ by the $X_i$’s and

$$d_P((t, x), (s, y)) = (d(x, y)^2 + |t-s|)^{1/2}$$

is its ‘parabolic’ counterpart in $\mathbb{R}^{n+1}$, we will assume that the $a_{ij}$’s are Hölder continuous on $\mathbb{R} \times \Omega$ with respect to the distance $d_P$.

One of the main motivations of the present Note is to provide a linear framework for the aforementioned fully nonlinear equations, by performing a deep analysis of the general class of Hörmander heat-type operators (2).

Our main results, proved under the above assumptions, can be briefly summarized as follows. We shall prove the existence and basic properties of a fundamental solution $h_A$ for the operator $H_A$, including representation formulas for the solution to the Cauchy problem. Also, we will show that $h_A$ satisfies a $C^2_{\text{loc}}$-regularity estimate, far off the pole, which means that $h_A$ and its derivatives $X_i h_A$, $X_j h_A$, $\partial_{ij} h_A$ are locally $d_P$-Hölder continuous. These results are contained in Theorem 1.1 below. Strictly related to the proof of existence of $h_A$, and of independent interest, are several sharp Gaussian bounds for $h_A$ that we will establish (see Theorem 1.2). A remarkable consequence of these bounds is an invariant Harnack inequality for $H_A$ (see Theorem 1.3).

Before going on, a clarification is in order. The operator $H_A$ is initially assumed defined only on a cylinder $\mathbb{R} \times \Omega$ for some bounded $\Omega$. We will extend $H_A$ to the whole space $\mathbb{R}^{n+1}$, in such a way that, outside a compact spatial set, it coincides with the classical heat operator. Henceforth all our statements will be referred to this extended operator.

**Theorem 1.1 (Existence of a fundamental solution).** Under the above assumptions, there exists a global fundamental solution $h_A(t, x; \tau, \xi)$ for $H_A$ in $\mathbb{R}^{n+1}$, with the properties listed below.

(i) $h_A$ is a continuous function away from the diagonal of $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$; $h_A(t, x; \tau, \xi) = 0$ for $t \leq \tau$. Moreover, for every fixed $\xi \in \mathbb{R}^{n+1}$, $h_A(\cdot, \xi) \in C_{\text{loc}}^{2,\alpha}(\mathbb{R}^{n+1} \setminus \{\xi\})$, and we have

$$H_A h_A(\cdot, \xi) = 0 \quad \text{in} \quad \mathbb{R}^{n+1} \setminus \{\xi\}.$$

(ii) For every $\psi \in C_0^\infty(\mathbb{R}^{n+1})$, the function $w(z) = \int_{\mathbb{R}^{n+1}} h_A(z; \xi) \psi(\xi) \, d\xi$ belongs to the class $C_{\text{loc}}^{2,\alpha}(\mathbb{R}^{n+1})$, and we have

$$H_A w = \psi \quad \text{in} \quad \mathbb{R}^{n+1}.$$

(iii) Let $\mu \geq 0$ and $T_2 > T_1$ be such that $(T_2 - T_1)\mu$ is small enough. Then, for every $f \in C^\beta([T_1, T_2] \times \mathbb{R}^n)$ (where $0 < \beta \leq \alpha$) and $g \in C(\mathbb{R}^n)$ satisfying the growth condition $|f(x, t)|, |g(x)| \leq c \exp(\mu d(x, 0)^2)$ for some constant $c > 0$, the function

$$u(x, t) = \int_{\mathbb{R}^n} h_A(t, x; T_1, \xi) g(\xi) \, d\xi + \int_{[T_1, t] \times \mathbb{R}^n} h_A(t, x; \tau, \xi) f(\tau, \xi) \, d\tau \, d\xi, \quad x \in \mathbb{R}^n, \ t \in (T_1, T_2],$$

belongs to the class $C_{\text{loc}}^{2,\beta}((T_1, T_2) \times \mathbb{R}^n) \cap C([T_1, T_2] \times \mathbb{R}^n)$. (We explicitly remark that, under our assumptions, $d(x, 0)$ is equivalent to the Euclidean norm $|x|$, for $|x| > 1$.) Moreover, $u$ is a solution to the following Cauchy problem

$$H_A u = f \quad \text{in} \quad (T_1, T_2) \times \mathbb{R}^n, \quad u(\cdot, T_1) = g \quad \text{in} \quad \mathbb{R}^n.$$

**Theorem 1.2 (Gaussian bounds).** There exists a positive constant $M$ and, for every $T > 0$, there exists a positive constant $c = c(T)$ such that, for $0 < t - \tau \leq T$, $x, \xi \in \mathbb{R}^n$, the following estimates hold.
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Following the general strategy used in the case of homogeneous groups in [1,2], our study proceeds in two steps. First we shall consider operators of kind (2) with constant coefficients $a_{ij}$. For these operators, existence and basic properties of the fundamental solution are guaranteed by known results. Here the point is to prove sharp Gaussian bounds on $h_A$ of the kind stated in Theorem 1.2 (but with bounds on derivatives of any order), which have to be uniform in the ellipticity class of the matrix $A$. To prove these uniform bounds, we have followed as close as possible...
the techniques of [8], the main new difficulties being the following: first, we have to take into account the dependence on the matrix $A$, getting estimates depending on $A$ only through the number $\lambda$; second, our estimates have to be global in space, while in [8] the authors work on a compact manifold; third, we need estimates on the difference of the fundamental solutions of two operators which have no analogue in [8]. The procedure is technically involved and a crucial role is played by the Rothschild–Stein lifting theorem [11].

The second step consists in studying operators with variable Hölder continuous coefficients $a_{ij}$, and applying the results about constant coefficient operators to establish existence and Gaussian bounds for the fundamental solution of those operators. This is accomplished by a suitable adaptation of the classical Levi’s parametric method. Regularity of $C^{2,\alpha}_{\text{loc}}$-type for the fundamental solution is then proved, applying the Schauder estimates for operators of kind (2), recently proved by two of us (see [5]). Finally, thanks to Theorems 1.1 and 1.2, the proof of Harnack inequality for $H_A$ can follow the lines drawn in [3]. The main tool here is a suitable adaptation of Krylov and Safanov’s methods, already used by Fabes and Stroock [7] in the classical parabolic case, and by Kusuoka and Stroock [9] for the operators in (2) with coefficients $a_{ij} = \delta_{ij}$. However, we do not use any of the probabilistic techniques and results used in [9]. Our approach is based on the existence of the relevant Green functions on a suitable family of cylindrical open sets, which are regular for the parabolic boundary value problems. We also use a noninvariant Harnack inequality for Hörmander operators proved by Bony in [4].

All the proofs of the results announced above will appear in [6].

References