L^p estimates for some ultraparabolic operators with discontinuous coefficients

M. Bramanti (1), **M. C. Cerutti** (1), **M. Manfredini** (2)

(1): Dipartimento di Matematica, Politecnico di Milano.
Via Bonardi 9. 20133 Milano. Italy.

(2): Dipartimento di Matematica, Università di Bologna.
Piazza Porta S. Donato 5. 40126 Bologna. Italy.

Keywords: hypoelliptic operators; singular integrals.
MOS (AMS) Subject Classifications: 35H05; 35R05; 43A85.

This is the text of a conference held by the first Author at the "Sixth International Colloquium on Differential Equations", Plovdiv, 18-23 August 1995. We refer to [BCM] *for details.*

0. INTRODUCTION

The subject of this lecture is related with two different fields of Analysis, namely the theory of a priori estimates for partial differential operators -in this case of hypoelliptic type- and the theory of singular integrals on homogeneous spaces.

To introduce and motivate the problem that we will discuss, let us consider the following linear partial differential operator:

$$L \equiv \sum_{i,j=1}^{q} a_{ij}\, \partial_{x_i x_j} + \sum_{i,j=1}^{N} b_{ij}\, x_i\, \partial_{x_j} - \partial_t \ \equiv$$

$$\equiv \sum_{i,j=1}^{q} a_{ij}\, \partial_{x_i x_j} + \langle x, BD\rangle - \partial_t\,, \qquad D = (\partial_{x_1}, \partial_{x_2}, \dots, \partial_{x_N}) \tag{0.1}$$

where $(x,t) \in \mathbb{R}^{N+1}$, $\{a_{ij}\}$ is a constant, symmetric, positive matrix $q \times q$, that is the principal part is uniformly elliptic on $\mathbb{R}^q$ ($q \le N$), and B is an $N \times N$ constant matrix. If $q < N$, the operator is *ultraparabolic*; therefore solutions to the equation $Lu = 0$, for instance, can have very bad regularity properties. However, if the matrix B has a suitable structure (that will be specified later), it turns out that the operator L is *hypoelliptic*: this means, by definition, that if $Lu \in C^\infty(U)$ for some open set $U \subseteq \mathbb{R}^N$, then $u \in C^\infty(U)$.

Let us recall a famous theorem proved by Hörmander in 1967 (see [H]):

Theorem 0.1. Let $L = \sum_{i=1}^{q} X_i^2 + X_0$, where $X_i = \sum_{j=1}^{N} c_{ij}(x)\, \partial_{x_j}$ ($c_{ij} \in C^\infty(\mathbb{R}^N)$, $i = 0, \dots, q$).

Then L is hypoelliptic if and only if the vector fields X_i satisfy the following condition:

$$\text{Rank Lie } \langle X_0, X_1, \dots, X_q\rangle = N \quad \text{at every point of } \mathbb{R}^N. \tag{0.2}$$

In other words, condition (0.2) means that, if we consider the vector fields X_i together with their commutators $[X_i, X_j]$, $[[X_i, X_j], X_k], \dots$, (until a certain step), then these fields span the whole tangent space $\mathbb{R}^N$, at every point.

Operators of the form (0.1) have been recently studied by Lanconelli and Polidoro (see [LP]), who proved that, for these operators, the following conditions are equivalent:

L is hypoelliptic; (0.3.a)
the vector fields $X_i = \partial_{x_i}$ ($i = 1, \dots, q$), $X_0 = \langle x, BD\rangle$ satisfy condition (0.2); (0.3.b)
the matrix B has the following structure: (0.3.c)

$$B \equiv \begin{bmatrix} * & B_1 & 0 & \dots & 0 \\ * & * & B_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \dots & B_r \\ * & * & * & \dots & * \end{bmatrix}$$

where for every $k=1,\dots,r$, B_k is a matrix $p_{k-1}\times p_k$ with rank p_k and $q=p_0\geq p_1\geq\dots\geq p_r$, $p_0+p_1+\dots+p_r=N$; the *'s stand for completely arbitrary elements.

From now on, we will always refer to operators of the form (0.1) satisfying conditions (0.3). Under these assumptions, the matrix B induces in the space $\mathbb{R}^{N+1}$ a (noncommutative) *group of translations*

$$(x,t)\circ(y,\tau)=(y+E(\tau)x,t+\tau),\quad \text{where } E(\tau)=\exp(-\tau B^T), \tag{0.4}$$

such that the operator L is invariant for left-translations. Moreover, if all the elements * in (0.3.b) are zero, there exists also a *group of dilations* on $\mathbb{R}^{N+1}$, which we denote by $\left\{D(\lambda)\right\}_{\lambda>0}$. Every $D(\lambda)$ is a group homomorphism, acting as follows:

$$D(\lambda)(x,t)=(\lambda^{\alpha_1}x_1,\dots,\ \lambda^{\alpha_N}x_N,\lambda^2 t) \tag{0.5}$$

where α_i are suitable integers ≥ 1. The operator L is homogeneous of degree 2 with respect to the group of dilations:

$$L\left(u(D(\lambda)z)\right)=\lambda^2\cdot D(\lambda)\left((Lu)(z)\right). \tag{0.6}$$

Let us point out some examples of the structures described above. (See also [LP] for further references).

***Example* 0.2.** (General case: there exists only the group of translations). Fokker-Planck operators in $\mathbb{R}^{2n+1}$:

let $B=\begin{bmatrix}0 & I\\ I & 0\end{bmatrix}$, where I is the identity matrix on $\mathbb{R}^n$, that is:

$$L\equiv\sum_{i,j=1}^{n}a_{ij}\,\partial_{x_ix_j}+\sum_{i=1}^{n}\left(x_i\,\partial_{x_{n+i}}+x_{n+i}\,\partial_{x_i}\right)-\partial_t\,.$$

In the simplest case, $n=1$, the operator is:

$$L=\partial^2_{x_1}+\left(x_1\partial_{x_2}-x_2\partial_{x_1}\right)-\partial_t$$

and the group of translations can be easily computed:

$$(x_1,x_2,t)\circ(y_1,y_2,\tau)=(y_1+x_1Ch\tau-x_2Sh\tau,y_2-x_1Sh\tau+x_2Ch\tau,t+\tau).$$

***Example* 0.3.** (Special case: the elements * are zero, and there exist both the group of translations and the group of dilations). Kolmogorov operators in $\mathbb{R}^{2n+1}$:

let $B=\begin{bmatrix}0 & I\\ 0 & 0\end{bmatrix}$, where I is the identity matrix on $\mathbb{R}^n$, that is:

$$L\equiv\sum_{i,j=1}^{n}a_{ij}\,\partial_{x_ix_j}+\sum_{i=1}^{n}x_i\,\partial_{x_{n+i}}-\partial_t\,.$$

For $n=1$, the operator is:

$$L=\partial^2_{x_1}+\ x_1\partial_{x_2}-\partial_t\,,$$

and the groups of translations and dilations are, respectively:

$$(x_1,x_2,t)\circ(y_1,y_2,\tau)=(x_1+y_1,x_2+y_2-\tau x_1,t+\tau);$$
$$D(\lambda)\ (x_1,x_2,t)=(\lambda x_1,\lambda^3x_2,\lambda^2t).$$

1. STATEMENT OF THE PROBLEM AND MAIN RESULT

Let us consider now the following class of operators with variable coefficients, modeled on the class of those operators of the form (0.1) for which there exist both a group of translations and a group of dilations. With the same notations of the previous section, and $z=(x,t)$, let:

$$L\equiv\sum_{i,j=1}^{q}a_{ij}(z)\partial_{x_ix_j}+\langle x,BD\rangle-\partial_t. \tag{1.1}$$

We are interested in proving local $\mathcal{L}^p$-estimates for the (appropriate) derivatives of solutions to the equation $Lu=f$. More precisely, let us introduce the following "Sobolev norm":

$$\| u \|^p_{S^p(L,\Omega)} = \| u \|^p_{\mathcal{L}^p(\Omega)} + \sum_{i=1}^{q} \| u_{x_i} \|^p_{\mathcal{L}^p(\Omega)} + \sum_{i,j=1}^{q} \| u_{x_i x_j} \|^p_{\mathcal{L}^p(\Omega)} + \| Yu \|^p_{\mathcal{L}^p(\Omega)} \tag{1.2}$$

where $Yu = \langle x, BDu \rangle - \partial_t u$ and Ω is any bounded domain in $\mathbb{R}^{N+1}$.

Our assumptions on L will be the following.

(i) The matrix $a_{ij}(z)$ is symmetric, bounded, measurable and uniformly elliptic on $\mathbb{R}^q$: there exists $\mu > 0$ such that

$$\frac{1}{\mu} \sum_{i=1}^{q} \xi_i^2 \leq \sum_{i,j=1}^{q} a_{ij}(z) \xi_i \xi_j \leq \mu \sum_{i=1}^{q} \xi_i^2 \tag{1.3}$$

for every $z \in \mathbb{R}^{N+1}$ and $(\xi_1, \xi_2, \ldots, \xi_q) \in \mathbb{R}^q$.

(ii) The matrix $B = \{b_{ij}\}$ has the form (0.3.c), with all the *'s equal to zero.

Note that under conditions (i)-(ii), for any fixed $z_0 \in \mathbb{R}^{N+1}$, the "frozen" operator

$$L_{z_0} = \sum_{i,j=1}^{q} a_{ij}(z_0) \partial_{x_i x_j} + \sum_{i,j=1}^{N} b_{ij} \, x_i \, \partial_{x_j} - \partial_t \tag{1.4}$$

is hypoelliptic. The last assumption we need on the coefficients a_{ij} is:

(iii) $a_{ij} \in VMO(\mathbb{R}^{N+1}, L)$. Here "$VMO$" stands for "*vanishing mean oscillation*"; the space VMO has been introduced by Sarason [Sa] and is defined as follows. $f \in VMO$ if and only if f is a locally integrable function such that

$$\eta_f(r) = \sup_{\rho < r} ⨍_{B_\rho} | f(z) - f_{B_\rho} | \, dz \to 0 \quad \text{for} \quad r \to 0. \tag{1.5}$$

Here $\mathcal{B}_\rho$ is a ball of radius ρ, with respect to a suitable distance, related to L, which will be introduced in section 2, and $⨍_{\mathcal{B}_\rho} f = f_{B_\rho}$ is the average of f over $\mathcal{B}_\rho$.

Our main result is the following:

Theorem 1.1. (See [BCM]). If (i), (ii), (iii) hold, $1 < p < \infty$, $\Omega' \subset \subset \Omega \subset \mathbb{R}^{N+1}$, ($\Omega$, Ω' bounded open sets), then there exists a positive constant c such that

$$\| u \|_{S^p(L,\Omega')} \leq c \left(\| Lu \|_{\mathcal{L}^p(\Omega)} + \| u \|_{\mathcal{L}^p(\Omega)} \right) \quad \text{for every } u \in S^p(L, \Omega). \tag{1.6}$$

The constant c depends only on p, μ, Ω', the matrix B (both through the entries b_{ij} and through the numbers p_i) and the "VMO moduli" η of the coefficients a_{ij}.

In the case of uniformly elliptic operators in nondivergence form with VMO coefficients, local and global $\mathcal{L}^p$-estimates have been proved by Chiarenza, Frasca and Longo, see [CFL1], [CFL2]. Since both the space of uniformly continuous functions and the Sobolev space $W^{1,N}$ (where N is the dimension) are properly contained in the class VMO, the above results improve the classical $\mathcal{L}^p$-theory by Agmon-Douglis-Nirenberg [ADN] as well as the results by Miranda [M]. The case of uniformly parabolic operators in nondivergence form, with time-dependent coefficients belonging to the (parabolic) VMO class has been treated by Bramanti and Cerutti in [BC1]. In this case the $\mathcal{L}^p$-theory extends the classical one developed in the sixties by Solonnikov [So] and, independently, by Fabes [Fa]. Therefore, the present research can be seen as a further step in the direction of the above results. However, in the hypoelliptic case the *geometry* related to the differential operator plays a major role, as we shall see.

Before going on, let us mention some examples showing what kinds of discontinuities are allowed for VMO functions (see [B] for details). The function

$$f(x) = \sin \log | \log | x | |$$

is a bounded, discontinuous function belonging (in a neighbourhood of the origin) to VMO. The same is true for

$$f(x) = \sin \left(| \log | x | |^\alpha \right)$$

for every $\alpha \in (0{,}1)$. Moreover, if $\alpha \geq 1 - 1/n$, this function does *not* belong to $W^{1,N}$.

2. STRUCTURE OF "HOMOGENEOUS SPACE" INDUCED IN $\mathbb{R}^{N+1}$ BY THE MATRIX B

To illustrate the strategy of the proof of $\mathcal{L}^p$-estimates, we first have to introduce in $\mathbb{R}^{N+1}$ some structures related to the group of translations and the group of dilations that we have defined in section 0 (see (0.4)-(0.5)).

1. The "*homogeneous norm*". For any $z \in \mathbb{R}^{N+1}$, $z \neq 0$, let us define

$$\| z \| = \rho \quad \Leftrightarrow \quad | D(\tfrac{1}{\rho})z | = 1,$$

with $| \cdot | =$ euclidean norm; also, let $\| 0 \| = 0$. Then:

$\| D(\lambda)z \| = \lambda \| z \|$ for every $z \in \mathbb{R}^{N+1}$, $\lambda > 0$;
The set $\{z \in \mathbb{R}^{N+1}\colon \| z \| =1\}$ is the euclidean sphere $\sum_{N+1}$;
For every $z, \zeta \in \mathbb{R}^{N+1}$

(i) $$\| z+\zeta \| \leq \| z \| + \| \zeta \| ;$$

(ii) $$\| z \circ \zeta \| \leq c(\| z \| + \| \zeta \|) \quad \text{for some constant } c = c(B) \geq 1.$$

2. The "*quasidistance*" d. In view of the above properties, it is natural to define:

$$d(z,\zeta) = \| \zeta^{-1} \circ z \| .$$

For d the following hold:

$$d(z,\zeta) \geq 0,\ d(z,\zeta)\text{=0 if and only if } z=\zeta;$$

$$\tfrac{1}{c}\, d(\zeta,z) \leq d(z,\zeta) \leq c\, d(\zeta,z) \quad \text{and} \quad d(z,\zeta) \leq c \left(d(z,z') + d(z',\zeta)\right)$$

for every $z, \zeta, z' \in \mathbb{R}^{N+1}$, some positive constant $c = c(B)$.

We define the balls with respect to *d*:

$$\mathcal{B}(z,r) \equiv \mathcal{B}_r(z) \equiv \{\zeta \in \mathbb{R}^{N+1}\colon d(z,\zeta) < r\}.$$

Note that $\mathcal{B}(0,r)$=$D(r)\mathcal{B}(0,1)$.

Now, let us consider the relationship between Lebesgue's measure and the above structures:

3. The Lebesgue's measure is invariant under the mappings:

$$z \mapsto z \circ \zeta \quad (\zeta \text{ fixed});$$

$$z \mapsto \zeta \circ z \quad (\zeta \text{ fixed});$$

$$z \mapsto z^{-1}.$$

4. $$det\, D(\lambda) = \lambda^{Q+2} \tag{1.2}$$

where $Q = \alpha_1 + \ldots + \alpha_N$, with α_i as in (0.5). We will call $Q+2$ the *homogeneous dimension* of $\mathbb{R}^{N+1}$ with respect to $(D(\lambda))_{\lambda>0}$. We will also call Q the *spatial homogeneous dimension* of $\mathbb{R}^{N+1}$ with respect to $(D(\lambda))_{\lambda>0}$.

From 3-4 it follows that:

5. $| \mathcal{B}(z,r) | = | \mathcal{B}(0,r) | = | \mathcal{B}(0,1) |\, r^{Q+2}$, for every $z \in \mathbb{R}^{N+1}$ and $r > 0$.

This implies that dz is a *doubling measure* with respect to d, that is

$$| \mathcal{B}(z,2r) | \leq c \cdot | \mathcal{B}(z,r) | \quad \text{for every } z \in \mathbb{R}^{N+1} \text{ and } r > 0$$

and therefore the space ($\mathbb{R}^{N+1}$,dz, d) is a *homogeneous space* in the sense of Coifman-Weiss (see [CW]).

3. STRATEGY OF THE PROOF OF $\mathcal{L}^p$-ESTIMATES

The proof of Theorem 1.1 follows, as close as possible, the same line of the elliptic and parabolic cases (see [CFL1,2], [BC1]). We will sketch the four steps of the proof, emphasizing the main differences with the above cases.

1^{st} STEP: REPRESENTATION FORMULA AND SINGULAR INTEGRALS

(i) We consider the "frozen" operator L_{z_0} (see (1.4)); by [LP], there exists a fundamental solution $\Gamma(z_0\,;\cdot\,)$ such that, for every test function u,

$$u(z) \;=\; -\int_{\mathbb{R}^{N+1}} \Gamma(z_0\,;\zeta^{-1}\circ z)\, L_{z_0}u(\zeta)d\zeta. \tag{3.1}$$

(ii) The function $\Gamma(z_0\,;\cdot\,)$ has the following properties:

$\Gamma(z_0\,;\cdot\,) \in C^\infty(\mathbb{R}^{N+1}\backslash\{0\})$;

$\Gamma(z_0\,;\cdot\,)$ is $D(\lambda)$-homogeneous of degree $-Q$;

the kernel $\Gamma_{ij}(z_0\,;z) \equiv \partial_{x_ix_j}[\Gamma(z_0\,;\cdot\,)](z)$ is $D(\lambda)$-homogeneous of degree $-(Q+2)$ and satisfies the vanishing property:

$$\int_{r<\|z\|<R} \Gamma_{ij}(z_0\,;z)\,dz = 0 \quad \text{for every } 0<r<R<\infty.$$

(iii) Using the above properties, we can differentiate twice (3.1); writing $L_{z_0} = L + (L_{z_0} - L)$ and letting finally z be equal to z_0, we get the following representation formula:

Theorem 3.1. Let $u \in C_0^\infty(\mathbb{R}^{N+1})$, $u = 0$ for $t \le 0$, $z \in \text{sprt}\, u$. Then, for $i, j = 1, \dots, q$,

$$u_{x_ix_j}(z) \;=\; -\lim_{\epsilon\to 0}\int_{\|\zeta^{-1}\circ z\|\ge\epsilon}\Gamma_{ij}(z;\zeta^{-1}\circ z)\left(\sum_{h,k=1}^{q}[a_{hk}(z)-a_{hk}(\zeta)]\,u_{x_hx_k}(\zeta) + \right.$$

$$\left. +\; Lu(\zeta)\right)d\zeta \;-\; Lu(z)\cdot\int_{\|\zeta\|=1}\Gamma_j(z\,;\zeta)\,\nu_i\,d\sigma(\zeta) \tag{3.2}$$

where ν_i is the i-th component of the outer normal to the surface Σ_{n+1}.

In order to rewrite the above formula in a more compact form, let us introduce the following singular integral operators:

$$K_{ij}f(z) = P.V.\int \Gamma_{ij}(z;\zeta^{-1}\circ z)\,f(\zeta)\,d\zeta. \tag{3.3}$$

Moreover, for an operator K and a function $a \in \mathcal{L}^\infty(\mathbb{R}^{N+1})$, define the *commutator*

$$C[K,a](f) = K(af) - a\cdot K(f). \tag{3.4}$$

Finally, set

$$\alpha_{ij}(z) = \int_{\|\zeta\|=1}\Gamma_j(z\,;\zeta)\,\nu_i\,d\sigma(\zeta). \tag{3.5}$$

Then (3.2) becomes

$$u_{x_ix_j} \;=\; -K_{ij}(Lu) + \sum_{h,k=1}^{q} C[K_{ij}\,,a_{hk}](u_{x_hx_k}) + \alpha_{ij}\cdot Lu \quad \text{for } i,j=1,\dots,q. \tag{3.6}$$

Now the desired $\mathcal{L}^p$-estimate on $u_{x_ix_j}$ depends on suitable singular integral estimates. Namely, we can prove the following:

Theorem 3.2. For every $p \in (1,\infty)$ there exists a positive constant $c = c(p,\mu,B)$ such that for every $a \in \mathcal{L}^\infty(\mathbb{R}^{N+1})$, $f \in \mathcal{L}^p(\mathbb{R}^{N+1})$, $i,j = 1,\dots,q$:

$$\|\, K_{ij}(f)\,\|_{\mathcal{L}^p(\mathbb{R}^{N+1})} \;\le\; c\;\|\; f\;\|_{\mathcal{L}^p(\mathbb{R}^{N+1})} \tag{3.7}$$

$$\| C[K_{ij}, a](f) \|_{\mathcal{L}^p(\mathbb{R}^{N+1})} \leq c \| a \|_* \| f \|_{\mathcal{L}^p(\mathbb{R}^{N+1})} \tag{3.8}$$

with $\| a \|_* = \sup_{\mathcal{B}} ⨍_{\mathcal{B}} | a(z) - a_{\mathcal{B}} | \, dz = BMO$ seminorm of a.

Estimate (3.8) can be localized at the following way:

if the function a belongs to VMO (and not only to BMO), then for every $\epsilon > 0$ there exists $r_0 > 0$, depending on ϵ and the VMO modulus of a, such that for every $r \in (0, r_0)$, sprt $f \subseteq \mathcal{B}_r$

$$\| C[K_{ij}, a](f) \|_{\mathcal{L}^p(\mathcal{B}_r)} \leq c(p, \mu, B) \cdot \epsilon \| f \|_{\mathcal{L}^p(\mathcal{B}_r)}. \tag{3.9}$$

Therefore, from (3.6), (3.7), (3.9) we have:

Theorem 3.3. For every $p \in (1, \infty)$ there exist $c = c(p, \mu, B)$ and $r_0 = r_0(p, \mu, \eta, B)$ such that if $u \in C_0^\infty(\mathbb{R}^{N+1})$, $u = 0$ for $t \leq 0$, sprt $f \subseteq \mathcal{B}_r$ with $0 < r < r_0$, then, for $i, j = 1, \ldots, q$

$$\| u_{x_i x_j} \|_p \leq c \left\{ \epsilon \cdot \sup_{h,k} \| u_{x_h x_k} \|_p + \| Lu \|_p \right\},$$

that is

$$\| u_{x_i x_j} \|_p \leq c \| Lu \|_p. \tag{3.10}$$

(Remember that η stands for the VMO moduli of the coefficients a_{hk}).

Finally, with standard techniques of cutoff function and interpolation inequalities, from (3.10) Theorem 1.1 follows.

2^{nd} STEP: REDUCTION OF SINGULAR INTEGRALS "WITH VARIABLE KERNEL" TO SINGULAR INTEGRALS "OF CONVOLUTION TYPE", BY MEANS OF EXPANSION IN SERIES OF SPHERICAL HARMONICS.

To prove estimates (3.7)-(3.8), we have to handle singular integrals of kind (3.3), which are *not* "of convolution type" because of the presence of the first variable z in the kernel, which comes from the variable coefficients in the principal part of the differential operator L. To bypass this difficulty, we can apply a standard technique of expansion in series of spherical harmonics. This idea dates back to Calderon-Zygmund [CZ], in the case of "standard" singular integrals, and has been adapted to kernels with mixed homogeneities by Fabes-Riviere [FR]. We briefly describe this technique.

Let

$$\left\{ Y_{km} \right\}_{\substack{k=1,\ldots,g_m \\ m=0,1,2,\ldots}}$$

be an orthonormal system of spherical harmonics in $\mathbb{R}^{N+1}$, complete in $\mathcal{L}^2(\Sigma_{N+1})$. For any fixed $z \in \mathbb{R}^{N+1}$, $\zeta \in \Sigma_{N+1}$, we can expand:

$$\Gamma_{ij}(z;\zeta) = \sum_{m=1}^{\infty} \sum_{k=1}^{g_m} c_{ij}^{km}(z) \, Y_{km}(\zeta) \quad \text{for } i, j = 1, \ldots, q. \tag{3.11}$$

If $\zeta \in \mathbb{R}^{N+1}$, let $\zeta' = D(\| \zeta \|^{-1})\zeta$; by (3.11) and homogeneity of Γ_{ij} we have

$$\Gamma_{ij}(z;\zeta) = \sum_{m=1}^{\infty} \sum_{k=1}^{g_m} c_{ij}^{km}(z) \frac{Y_{km}(\zeta')}{\|\zeta\|^{Q+2}} \quad \text{for } i, j = 1, \ldots, q. \tag{3.12}$$

Then

$$K_{ij}(f)(z) = \sum_{m=1}^{\infty} \sum_{k=1}^{g_m} c_{ij}^{km}(z) \, T_{km} f(z) \tag{3.13}$$

with

$$T_{km}f(z) = P.V.\int H_{km}(\zeta^{-1} \circ z)\, f(\zeta)\, d\zeta \tag{3.14}$$

and

$$H_{km}(z) = \frac{Y_{km}(z')}{\|z\|^{Q+2}}\ . \tag{3.15}$$

Note that singular integrals (3.14) are of convolution type.

3^{rd} STEP: ESTIMATES ON SINGULAR INTEGRALS "OF CONVOLUTION TYPE" AND THEIR COMMUTATORS.

Let us review some known results about $\mathcal{L}^p$ continuity of singular integral operators, in relation with the theory of P.D.O. If L is a uniformly elliptic operator, the corresponding singular integrals are invariant with respect to euclidean translations, and homogeneous with respect to euclidean dilations. The suitable $\mathcal{L}^p$ estimates follow from classical Calderon-Zygmund theory, [CZ]. If L is a uniformly parabolic operator, euclidean dilations must be replaced with "parabolic" dilations. The theory of singular integrals with mixed homogeneities has been developed by Fabes-Riviere [FR]. If L is a hypoelliptic operator, written for instance as a sum of squares of vector fields, which is invariant with respect to a suitable group of translations and is homogeneous of degree two with respect to a group of dilations, the corresponding singular integrals are also $\mathcal{L}^p$ continuous; this theory has been developed by Folland [Fo]. All these results (at least in the convolution-type case) can be seen as particular cases of a general theorem about singular integrals on homogeneous spaces.

The theory of singular integrals on homogeneous spaces began to be developed around '70, with the works by Coifman-De Guzman [CG], Coifman-Weiss [CW], Korányi-Vagi [KV1, KV2], Rivière [R]. The first deep result in this direction, proved for instance in [CW], is the following: if a singular integral operator is $\mathcal{L}^2$-$\mathcal{L}^2$ continuous and its kernel satisfies a suitable inequality, then it is $\mathcal{L}^p$-$\mathcal{L}^p$ continuous for every $p \in (1, \infty)$. However, proving $\mathcal{L}^2$ continuity of a singular integral operator in the abstract context of homogeneous spaces, is a much harder problem than in $\mathbb{R}^N$, where we can use the Fourier's transform. In '84 David-Journée [DJ] proved (in $\mathbb{R}^N$) a new powerful criterium of $\mathcal{L}^2$ continuity, known as "$T(1)$-Theorem" (roughly speaking, the operator is $\mathcal{L}^2$ continuous if it maps the constant function 1 in BMO); in '85 this result was improved by David-Journée-Semmes [DJS] (with their "$T(b)$-Theorem"); in the same paper, they also generalized $T(1)$-Theorem to homogeneous spaces. In '90, Christ [Ch] obtained another, more general, proof of $T(1)$-Theorem in homogeneous spaces. Using this theorem and the results of [CW], it is easy to prove $\mathcal{L}^p$-continuity of a singular integral whose kernel satisfies "standard" assumptions. See [BC2] for details.

The context of homogeneous spaces is also the natural one where an estimate of kind (3.8) on the commutator of a singular integral operator with a *BMO* function can be proved. This result was obtained by Coifman-Rochberg-Weiss [CRW] in the euclidean case, and extended by Bramanti-Cerutti [BC2] to homogeneous spaces. This extension is not straightforward, because of the possibility for a homogeneous spaces of having finite measure: in this case new phenomena must be taken into account, and the statement of the result is a little more involved. Since we will apply this theorem to a space of infinite measure, we state it in this particular (easier) case.

Definition 3.4. We say that K is a Calderón-Zygmund operator (*CZ* operator) on the homogeneous space (X, d, μ) if:

i) K:$\mathcal{L}^p(X) \to \mathcal{L}^p(X)$ is linear continuous for every $p \in (1, \infty)$;

ii) there exists a measurable function k:$X \times X \to \mathbb{R}$ such that for every "test function" f (that is, f is measurable, bounded, and has bounded support) for a.e. $x \notin sprt f$:

$$Kf(x) \;=\; \int_X k(x,y) f(y)\, d\mu(y)\ ;$$

iii) the kernel k satisfies the following "pointwise Hörmander condition":

there exist constants $c_K > 0$, $\beta > 0$, $M > 1$ such that for every $x_0 \in X$, $r > 0$, $x \in B_r(x_0)$, $y \notin B_{Mr}(x_0)$,

$$|\ k(x_0,y) - k(x,y)\ | \;\le\; \frac{c_K}{\mu(B(x_0;y))} \cdot \frac{d(x_0,x)^\beta}{d(x_0,y)^\beta} \tag{3.16}$$

where, $B(x;y)$ is the ball with center x and radius $d(x,y)$. We will write $c(p,K)$ for a constant depending on the number $p \in (1, \infty)$, the $\mathcal{L}^p$ norm of K and the constants in (3.16).

Theorem 3.5. (See [BC2]). Let (X, d, μ) be a homogeneous space with $\mu(X) = \infty$, K a *CZ* operator, $a \in BMO$. Then for every $p \in (1, \infty)$ the operator $C[K, a]$ can be continuously extended to $\mathcal{L}^p$, and

$$\|\ C[K,a]\ \|_{\,p} \;\le\; c(p,K,X)\ \|\ a\ \|_{\,*}\ \|\, f\,\|_{\,p}.$$

Theorem 3.5 applies to the commutators of the "convolution-type" operators T_{km} (see (3.12)-(3.15)).

4^{th} STEP: CONVERGENCE OF THE SERIES.

So far, we have proved $\mathcal{L}^p$ estimates for T_{km} and $C[T_{km}, a]$. We recall expansion (3.13):

$$K_{ij}(f)(z) = \sum_{m=1}^{\infty} \sum_{k=1}^{g_m} c_{ij}^{km}(z)\, T_{km} f(z)$$

Then, to prove Theorem 3.2 (and so our final result) we need:
1) explicit dependence of the constants appearing in $\mathcal{L}^p$ estimates on the indexes k, m;
2) $\mathcal{L}^\infty$ bounds on the coefficients c_{ij}^{km}.

Point (1) follows from the proof of Theorem 3.5 and known properties of spherical harmonics, which are involved in the definition of T_{km}. Point (2) is much more delicate. By known properties of spherical harmonics, to get this bound we need to know that, for every multi-index β,

$$\sup_{\substack{\| \zeta \| = 1 \\ z \in \mathbb{R}^{N+1}}} \left| \left(\frac{\partial}{\partial \zeta} \right)^{\beta} \Gamma_{ij}(z;\zeta) \right| \le c(\beta). \tag{3.17}$$

Estimate (3.17) is a uniform bound on the derivatives of a family of fundamental solutions. In the elliptic and parabolic cases (see [CFL1], [BC1]), this follows from the explicit formula of the fundamental solution, by uniform ellipticity. In our case, properties of the fundamental solutions have been studied by [LP], who also refer to [H], [K]. They prove the following explicit formula for Γ:

$$\Gamma(z_0; z) = \frac{t^{-Q/2}}{(4\pi)^{N/2} (\det C(z_0))^{1/2}} \exp\left(-\frac{1}{4} \langle C^{-1}(z_0) D_0(\tfrac{1}{\sqrt{t}}) x, D_0(\tfrac{1}{\sqrt{t}}) x \rangle \right) \tag{3.18}$$

with $z = (x, t)$ and $t > 0$. Here $D_0(\lambda)$ is the restriction to $\mathbb{R}^N$ of the group of dilations $D(\lambda)$; moreover

$$C(z_0) = \int_0^1 E(s) A(z_0) E^T(s) ds$$

where $E(s)$ is as in (0.4); $A(z_0)$ is the $N \times N$ constant matrix

$$A(z_0) = \begin{bmatrix} A_0(z_0) & 0 \\ 0 & 0 \end{bmatrix}$$

and $A_0(z_0)$ is the $q \times q$ matrix $A_0(z_0) = \left(a_{ij}(z_0) \right)_{i,j=1,\dots,q}$.

Moreover, by assumptions (i)-(ii) in section 1, $C(z_0)$ is positive, *uniformly in* z_0. By this result, proved in [LP], the uniform bound (3.17) follows from (3.18). This completes the proof of Theorem 3.2.

References

[ADN] S. Agmon-A. Douglis-L. Nirenberg: Estimates near the boundary for solutions of elliptic partial differential eqautions satisfying general boundary conditions. *I*: *Comm. Pure Appl. Math.,* 12 (1959), 623-727; *II*: *Comm. Pure Appl. Math.,* 17 (1964), 35-92.

[B] M. Bramanti: Commutatori di integrali singolari ed equazioni paraboliche a coefficienti discontinui. *Tesi di Dottorato di Ricerca in Matematica, Milano*, 1993.

[BC1] M. Bramanti-M. C. Cerutti: $W_p^{1,2}$-solvability for the Cauchy-Dirichlet problem for parabolic equations with VMO coefficients. *Comm. in Part. Diff. Eq.*, **18** (9&10) (1993), 1735-1763.

[BC2] M. Bramanti-M. C. Cerutti: Commutators of singular integrals in homogeneous spaces. To appear on *Boll. Unione Mat. Italiana.*

[BCM] M. Bramanti-M. C. Cerutti-M. Manfredini: $\mathcal{L}^p$-estimates for some ultraparabolic operators with discontinuous coefficients. To appear on: *Journal of Math. Anal. and Appl.*

[CFL1] F. Chiarenza-M. Frasca-P. Longo: Interior $W^{2,p}$-estimates for nondivergence elliptic equations with discontinuous coefficients. *Ricerche di Mat.* **XL** (1991), 149-168.

[CFL2] F. Chiarenza-M. Frasca-P. Longo: $W^{2,p}$*-solvability of the Dirichlet problem for non divergence elliptic equations with VMO coefficients. Trans. of Am. Math. Soc.*, 336 (1993), n. 1, 841-853.

[CG] R. Coifman-M. de Guzmán: Singular Integrals and Multipliers on Homogeneous Spaces. *Revista de la Unión Matemática Argentina*, 25 (1970), 137-143.

[Ch] M. Christ: Lectures on Singular Integral Operator. *CBMS Regional Conference Series in Mathematics, 77, A. M. S. Providence, RI, 1990.*

[CRW] R. Coifman-R. Rochberg-G. Weiss: Factorization theorems for Hardy spaces in several variables. *Annals of Math.*, 103 (1976) 611-635.

[CW] R. Coifman-G. Weiss: Analyse Harmonique Non-Commutative sur Certains Espaces Homogènes. *Lecture Notes in Mathematics*, n. **242**. Springer-Verlag, Berlin-Heidelberg-New York, 1971.

[CZ] A. P. Calderón-A. Zygmund: Singular integral operators and differential equations, *Am. Journ. of Math.*, **79** (1957), 901-921.

[DJ] G. David-J. L. Journé: A boundedness criterion for generalized Calderón-Zygmund operators. *Annals of Mathematics*, 120 (1984), 371-397.

[DJS] G. David-J. L. Journé-S Semmes: Opérateurs de Calderón-Zygmund, fonctions para-accrétives et interpolation. *Rev. Mat. Iberoamericana*, 1, n. 4 (1985), 1-56.

[Fa] E. B. Fabes: Singular integrals and partial differential equations of parabolic type. *Studia Mathematica*, 27 (1966), 19-38.

[Fo] G. B. Folland: Subelliptic estimates and function spaces on nilpotent Lie groups, *Arkiv for Math.* **13**, (1975), 161-207.

[FR] E. B. Fabes-N. Rivière: Singular integrals with mixed homogeneity. *Studia Mathematica*, **27**, (1966), 19-38.

[H] L. Hörmander: Hypoelliptic second order differential equations. *Acta Mathematica*, **119** (1967), 147-171.

[JN] F. John-L. Nirenberg: On functions of bounded mean oscillation. *Comm. Pure Appl. Math.*, 14 (1961), 175-188.

[K] L. P. Kuptsov: Fundamental solutions for a class of second-order elliptic-parabolic equations. *Differentcial'nye Uravnenija* **8** (1972), 1649-1660. English transl. *Differential Equations* **8** (1972), 1269-1278.

[KV1] A. Korányi-S. Vági: Intégrales singulières sur certains espaces homogènes. *C. R. Acad. Sc. Paris*, t. 268 (1969), *serie* A, 765-768.

[KV2] A. Korányi-S. Vági: Singular Integrals on Homogeneous Spaces and some Problems of classical analysis. *Ann. Sc. Norm. Sup. di Pisa, serie III, vol. XXV* (1971), pp. 575-648.

[LP] E. Lanconelli-S. Polidoro: On a class of hypoelliptic evolution operators, *Rend. Sem. Mat. Polit. Torino* **51.4** (1993), 137-171.

[M] C. Miranda: Sulle equazioni ellittiche del secondo ordine di tipo non variazionale a coefficienti discontinui. *Ann. Mat. Pura e Appl.*, 53 (1965), 353-386.

[P] S. Polidoro: Su una classe di operatori ultraparabolici di tipo Kolmogorov-Fokker-Planck, *Tesi di Dottorato di Ricerca in Matematica*, Università degli Studi di Bologna, 1993.

[R] N. M. Rivière: On singular integrals. *Bull. of A. M. S., vol.* 75 (1969), 843-847.

[RS] L. P. Rothschild-E. M. Stein: Hypoelliptic differential operators and nilpotent groups, *Acta Math.*, **137** (1976), 247-320.

[Sa] D. Sarason: Functions of vanishing mean oscillations. *Trans. of Am. Math. Soc.*, 207 (1975), 391-405.

[So] V. A. Solonnikov: A priori estimates for equations of second order of parabolic type. *Trudy Mat. Inst. Steklov* 70 (1964), 133-212, in russian; english translation: *Am. Math. Soc. Transl.* (2) 65 (1967), 51-137.

Milano, September '95.