Interior $HW^{1,p}$ estimates for divergence degenerate elliptic systems in Carnot groups

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1. Introduction

Let $X_1, \ldots, X_q$ be the basis of the space of horizontal vector fields on a homogeneous Carnot group $G = (\mathbb{R}^n, \circ)$ ($q < n$). We consider the following divergence degenerate elliptic system

$$
\sum_{\beta=1}^{N} \sum_{i,j=1}^{q} X_i (a^\beta_{ij}(x) X_j u^\beta) = \sum_{\alpha=1}^{N} X_i f^\alpha, \quad \alpha = 1, 2, \ldots, N
$$

where the coefficients $a^\beta_{ij}$ are real valued bounded measurable functions defined in $\Omega \subset G$, satisfying the strong Legendre condition and belonging to the space $VMO_{loc} (\Omega)$ (defined by the Carnot–Carathéodory distance induced by the $X_i$’s). We prove interior $HW^{1,p}$ estimates ($2 \leq p < \infty$) for weak solutions to the system.

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Moreover, the Lebesgue measure of integers, the Lie algebra of Carnot group and that for some fixed exponents $0 < \alpha < \beta$ as follows:

\[
\begin{align*}
\{ \text{'translation'}, \text{where the origin is the group identity,} \} \\
\text{and a family}
\end{align*}
\]

\[
\{ \text{of group automorphisms (''dilations''), acting}
\end{align*}
\]

\[
\text{on the solution to a constant coefficient system on a Carnot group is }
\]

\[
\text{known to satisfy an } L^\infty \text{ gradient bound (see Theorem 2.10) which turns out to be a key tool in our proof. This result about systems with constant coefficients in Carnot groups has been proved by Shores [31], and represents one of the main reasons why we have restricted ourselves to the case of Carnot groups instead of considering general Hörmander’s vector fields.}
\]

This paper represents the first case of study of $L^p$ estimates on the “subelliptic gradient” $Xu$ for subelliptic systems. Di Fazio and Fanciullo in [17] have deduced interior Morrey regularity in spaces $L^{p,\lambda}$ for weak solutions to the system (1.1) under the assumption that the coefficients $a^i_{\alpha\beta}$ belong to the class $VMO_N \cap L^\infty$, while Schauder-type estimates have been proved for subelliptic systems by Xu and Zuily [35].

This paper is organized as follows. In Section 2 we recall some basic facts about Carnot groups and state precisely our assumptions and main results; in Section 3 we prove the approximation result for local solutions to the original system by means of solutions to a system with constant coefficients; in Section 4 we prove some local estimates on the Hardy–Littlewood maximal function of $|Xu|^2$, and in Section 5 we come to the proof of our main result.

2. Preliminaries and statement of the results

2.1. Background on Carnot groups

We are going to recall here a few facts about Carnot groups that we will need in the following. For the proofs, more properties, and examples, we refer the reader to the paper [19], the books [1] and [32, Chapters XII–XIII].

**Definition 2.1 (Homogeneous Carnot Groups).** A homogeneous group $G$ is the set $\mathbb{R}^n$ endowed with a Lie group operation $\circ$ (“translation”), where the origin is the group identity, and a family $\{D(\lambda)\}_{\lambda > 0}$ of group automorphisms (“dilations”), acting as follows:

\[
D(\lambda) (x_1, x_2, \ldots, x_n) = (\lambda^{\alpha_1} x_1, \lambda^{\alpha_2} x_2, \ldots, \lambda^{\alpha_n} x_n) \quad \forall \lambda > 0
\]

for some fixed exponents $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n$. The number $Q = \sum_{j=1}^{n} \alpha_j$ is called the homogeneous dimension of $G$.

We say that a vector field $X = \sum_{j=1}^{n} a_j(x) \partial_{x_j}$ is left invariant if for any smooth function $f$ one has

\[
X^* (f \circ \lambda) = (Xf) (f \circ \lambda) \quad \forall x, y \in \mathbb{G};
\]

we say that $X$ is $k$-homogeneous if for any smooth function $f$ one has

\[
X (f (D(\lambda) x)) = \lambda^k (Xf) (D(\lambda) x) \quad \forall \lambda > 0, x \in \mathbb{G}.
\]

Let $X_i (i = 1, 2, \ldots, n)$ be the unique left invariant vector field on $G$ which at the origin coincides with $\partial_{x_i}$. We assume that for some integer $q < n$ the vector fields $X_1, X_2, \ldots, X_q$ are 1-homogeneous and satisfy Hörmander’s condition in $\mathbb{R}^n$: the Lie algebra generated by the $X_i$’s at any point has dimension $n$. Under these assumptions we say that $G$ is a homogeneous Carnot group and that $\{X_1, X_2, \ldots, X_n\}$ is the canonical basis of the space of horizontal vector fields.

The properties required in the above definition have a number of consequences: the exponents $\alpha_i$ are actually positive integers, the Lie algebra of $G$ is stratified, homogeneous and nilpotent; the vector fields $X_i$ have polynomial coefficients. Moreover, the Lebesgue measure of $\mathbb{R}^n$ is the Haar measure in $G$. 
Like for any set of Hörmander’s vector fields, it is possible to define the corresponding Carnot–Carathéodory distance $d_X$, as follows.

**Definition 2.2 (CC-distance).** For any $\delta > 0$, let $C_\delta$ be the set of absolutely continuous curves $\phi : [0, 1] \to \mathbb{R}^n$ such that

$$
\phi'(t) = \sum_{i=1}^q a_i(t) X_i(\phi_i(t)) \quad \text{with } |a_i(t)| \leq \delta \quad \text{for a.e. } t \in [0, 1].
$$

Then

$$
d_X(x, y) = \inf \{ \delta > 0 : \exists \phi \in C_\delta \text{ with } \phi(0) = x, \phi(1) = y \}.
$$

The function $d_X$ turns out to be finite for any couple of points, and is actually a distance, called Carnot–Carathéodory distance; due to the structure of Carnot group, $d_X$ is also left invariant and 1-homogeneous on $\mathbb{G}$. Let

$$
B_r(x) = \{ y \in \mathbb{G} : d_X(x, y) < r \}
$$

be the metric ball of center $x$ and radius $r$ in $\mathbb{G}$. Since the Lebesgue measure in $\mathbb{R}^n$ is the Haar measure on $\mathbb{G}$, one has (writing $|A|$ for the measure of $A$)

$$
|B_r(x)| = \omega_{\mathbb{G}} r^q,
$$

where $q$ is the homogeneous dimension of $\mathbb{G}$ and $\omega_{\mathbb{G}}$ is a positive constant.

Next, we need to define the function spaces we will use in the following.

**Definition 2.3 (Horizontal Sobolev Spaces).** For any $p \geq 1$ and domain $\Omega \subset \mathbb{G}$, let us define the *horizontal Sobolev space*:

$$
HW^{1,p}(\Omega; \mathbb{R}^N) = \{ u \in L^p(\Omega; \mathbb{R}^N) : \|u\|_{HW^{1,p}(\Omega; \mathbb{R}^N)} < \infty \}
$$

with the norm

$$
\|u\|_{HW^{1,p}(\Omega; \mathbb{R}^N)} = \|u\|_{L^p(\Omega; \mathbb{R}^N)} + \|Xu\|_{L^p(\Omega; \mathbb{R}^N)}
$$

having set

$$
\|u\|_{L^p(\Omega; \mathbb{R}^N)} = \|u\|_{L^p(\Omega)} \quad \text{with } |u| = \left( \sum_{\alpha=1}^N \sum_{i=1}^q |u^\alpha_i|^2 \right)^{1/2}
$$

and

$$
\|Xu\|_{L^p(\Omega; \mathbb{R}^N)} = \|Xu\|_{L^p(\Omega)} \quad \text{with } |Xu| = \left( \sum_{\alpha=1}^N \sum_{i=1}^q |Xu^\alpha_i|^2 \right)^{1/2}.
$$

Also, we define the space $HW^{1,p}_{loc}(\Omega; \mathbb{R}^N)$ as the space of functions $u$ such that $u \phi \in HW^{1,p}(\Omega; \mathbb{R}^N)$ for any $\phi \in C_0^\infty(\Omega)$ and the space $HW^{1,p}_0(\Omega; \mathbb{R}^N)$ as the closure of $C_0^\infty(\Omega; \mathbb{R}^N)$ in the norm $HW^{1,p}(\Omega; \mathbb{R}^N)$.

**Definition 2.4 (BMO-type Spaces).** For any $\Omega' \subset \Omega$, let $R_0$ be a number such that $B_r(x) \subset \Omega$ for any $x \in \Omega'$ and $r \leq R_0$. For any $f \in L^1_{loc}(\Omega)$ and $r \leq R_0$, let

$$
\eta_{\Omega'}(r) = \sup_{x_0 \in \Omega', 0 < r \leq R_0} \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} |f(x) - f_{B_r(x_0)}|^2\, dx,
$$

where

$$
f_{B_r(x_0)} = \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} f(x)\, dx.
$$

We say that $f$ is $(\delta, R)$-vanishing in $\Omega'$ (for a couple of fixed positive numbers $\delta, R$, with $R \leq R_0$) if

$$
\eta_{\Omega'}(R) < \delta^2.
$$

We say that $f \in VMO_{loc}(\Omega)$ if for any $\Omega' \subset \Omega$ and $R_0$ such that $B_r(x) \subset \Omega$ for any $r \leq R_0$ and $x \in \Omega'$, we have

$$
\eta_{\Omega'}(R) \to 0 \quad \text{as } r \to 0.
$$

The function $\eta_{\Omega'}(\cdot)$ is called the local VMO modulus of $f$ on $\Omega'$.

We will use the following well-known result by Jerison (see [25, Theorem 2.1] for the case $p = 2$ and [25, Section 6] for $p \neq 2$).
Theorem 2.5 (Poincaré’s Inequality). For $1 \leq p < \infty$ there exists a positive constant $c = c \left( \mathbb{G}, p \right)$, such that for any $u \in \text{HW}^{1,p}(B_R)$,

$$\|u - u_{B_R}\|_{L^p(B_R)} \leq cR \|Xu\|_{L^p(B_R)}.$$  \hfill (2.2)

If $u \in \text{HW}_0^{1,p}(B_R)$,

$$\|u\|_{L^p(B_R)} \leq cR \|Xu\|_{L^p(B_R)}.$$  \hfill (2.3)

The previous theorem holds for a general system of Hörmander’s vector fields; in that case, however, some restriction on the center and radius of the ball $B_R$ applies (see [25, Theorem 2.1]); on a Carnot group, instead, due to the dilation invariance of the inequalities (2.2) and (2.3), these hold for any ball $B_R$ and with an “absolute” constant $c$.

We will also make use of the following.

Definition 2.6 (Space of Homogeneous Type, See [16]). Let $S$ be a set and $d : S \times S \rightarrow [0, \infty)$ a quasidistance, that is, for some constant $c \geq 1$ one has

$$d \left( x, y \right) = 0 \iff x = y$$

$$d \left( x, y \right) = d \left( y, x \right)$$

$$d \left( x, y \right) \leq c \left[ d \left( x, z \right) + d \left( z, y \right) \right]$$

for all $x, y, z \in S$. The balls defined by $d$ induce a topology in $S$; let us assume that the $d$-balls are open in this topology. Moreover, assume that there exists a regular Borel measure $\mu$ on $S$, such that the “doubling condition” is satisfied:

$$\mu \left( B_2 \left( x \right) \right) \leq c\mu \left( B_1 \left( x \right) \right),$$

for every $r > 0$, $x \in S$ and some positive constant $c$. Then we say that $(S, d, \mu)$ is a space of homogeneous type.

Remark 2.7. Note that in our context any Carnot–Carathéodory ball $B_R \left( x_0 \right)$ is a $d_X$-regular domain (see for instance [4, Lemma 4.2]), that is there exists a positive constant $c_d$ such that

$$|B_R \left( x_0 \right) \cap B \left( x \right)| \geq c_d |B \left( x \right)| \forall r > 0, \forall x \in B_R \left( x_0 \right).$$

This implies that $(B_R \left( x_0 \right), d_X, dx)$ is a space of homogeneous type. Moreover, a simple dilation argument shows that, in a Carnot group, the constant $c_d$, and therefore the doubling constant of $(B_R \left( x_0 \right), d_X, dx)$, is independent of $R$.

2.2. Assumptions and known results about degenerate systems

The general assumptions which will be in force throughout the paper are collected in the following.

Assumption (H). We assume that $\mathbb{G}$ is a homogeneous Carnot group in $\mathbb{R}^n$ and $\{X_1, X_2, \ldots, X_q\}$ is the canonical basis of the space of horizontal vector fields in $\mathbb{G}$ (see Definition 2.1). We assume that the coefficients

$$\left\{ a_{i\alpha} \right\}_{i=1}^{\alpha=1}^{q},$$

in (1.1) are real valued, bounded measurable functions defined in $\Omega$ and satisfying the strong Legendre condition: there exists a constant $\mu > 0$ such that

$$\mu|\xi|^2 \leq a_{i\alpha}(x)\xi_i\xi_\alpha \leq \mu^{-1}|\xi|^2$$

for any $\xi \in \mathbb{R}^{N \times q}$, a.e. $x \in \Omega$.

We recall the standard definition of weak solution.

Definition 2.8. We say that $u \in \text{HW}^{1,2} \left( \Omega; \mathbb{R}^N \right)$ is a weak solution to the system (1.1), if it satisfies

$$\int_{\Omega} a_{i\alpha} \left( x \right) X^\alpha \varphi^\alpha dx = \int_{\Omega} f_i \varphi^\alpha dx$$

for any $\varphi \in \text{HW}_0^{1,2} \left( \Omega; \mathbb{R}^N \right)$.

Recall that on a Carnot group the transpose of a vector field is just the opposite: $X_i = -X_i$. Hence the above definition of weak solution is consistent with the way the system (1.1) is written.
Remark 2.9. Let $B_R \subset \Omega$ be any metric ball. If $f_{ij}^i \in L^2(B_R)$ and $u_0 \in H^{1.2}(b_R)$, then by assumption (2.7) and Poincaré’s inequality (2.3) we can apply Lax–Milgram’s theorem, and conclude that there exists a unique solution $u \in H^{1.2}(b_R; \mathbb{R}^N)$ to system (1.1) such that $u - u_0 \in H^{1.2}_\theta(b_R)$. Moreover, the following a priori estimate holds:

$$
\|u\|_{H^{1.2}(b_R; \mathbb{R}^N)} \leq C \left( \|F\|_{L^2(b_R;\mathbb{R}^{N\times q})} + \|u_0\|_{H^{1.2}(b_R; \mathbb{R}^N)} \right)
$$

for some constant $C$ only depending on $G$, $\mu$, $R$ (see [11, Chapter 8] for a proof of this fact in the elliptic case).

The next result is taken from [31, Corollary 19]. See also [23], where the analogous parabolic inequality is proved.

**Theorem 2.10.** Let $v \in H^{1.2}(B(x_0, KR); \mathbb{R}^N)$ be a solution to the system

$$
X_i \left( a_{ij}^\beta X_j v^\beta \right) = 0 \quad \text{in } B(x_0, KR)
$$

with constant coefficients $a_{ij}^\beta$ satisfying (2.7) and some $K > 1$. Then $v \in C^\infty(B(x_0, KR))$; moreover

$$
\sup_{B_R(x_0)} |Xv|^2 \leq C R^{-2} \frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} |v|^2 \, dx,
$$

where the positive constant $C$ depends on $K$, $\mu$, $G$, $N$ but is independent of $x_0$, $R$ and $v$.

The following result can be proved in a completely standard way by suitable cutoff functions (for the analogous elliptic version see for instance [11, Theorem 2.1 p.134]).

**Theorem 2.11** (Caccioppoli’s Inequality). Let $u \in H^{1.2}(B_R(x); \mathbb{R}^N)$ be a weak solution to (1.1) in $B_R(x) \subset \Omega$. There exists a constant $C > 0$ depending on $G$, $N$, $R$ such that for any $\rho \in (0, R)$,

$$
\int_{B_{\rho}(x)} |Xu(x)|^2 \, dx \leq C \left( \frac{1}{(R - \rho)^2} \int_{B_R(x)} |u(x)|^2 \, dx + \int_{B_{\rho}(x)} |F(x)|^2 \, dx \right).
$$

2.3. Statement of the result

We now state precisely the main result of this paper.

**Theorem 2.12.** Under Assumption (H), let the $a_{ij}^\beta$’s belong to $VMO_{loc}(\Omega)$ and let $\Omega' \subset \Omega$, $2 < p < \infty$. Then there is a positive constant $C$ depending on $G$, $\mu$, $p$, $\Omega$, $\Omega'$ and the local VMO moduli of the $a_{ij}^\beta$’s in $\Omega'$ such that if $F = (f_i^j) \in L^p(\Omega; \mathbb{M}^{N\times q})$ and $u \in H^{1.2}((\Omega'; \mathbb{R}^N)$ is a weak solution to (1.1) in $\Omega'$, then $u \in H^{1.2}(\Omega'; \mathbb{R}^N)$ and

$$
\|u\|_{H^{1.2}(\Omega'; \mathbb{R}^N)} \leq C \left( \|F\|_{L^p(\Omega';\mathbb{M}^{N\times q})} + \|u\|_{L^2(\Omega'; \mathbb{R}^N)} \right).
$$

In order to prove Theorem 2.12, we will prove the following local result.

**Theorem 2.13.** Under Assumption (H), for any $x \in \Omega$, $R_0 > 0$ such that $B_{11R_0}(x) \subset \Omega$ there exists $\delta = \delta(p, G, R_0, \mu) > 0$ such that for any $R \leq R_0$, if the coefficients $a_{ij}^\beta$ are $(\delta, 8R)$-vanishing in $B_R(x)$ and $p \in (2, \infty)$, then there is a positive $C = c(R, R_0, p, G)$ such that if $F = (f_i^j) \in L^p(B_{11R}(x); \mathbb{M}^{N\times q})$ and $u \in H^{1.2}(B_{11R}(x); \mathbb{R}^N)$ is a weak solution of (1.1) in $B_{11R}(x)$, then $u \in H^{1.2}(B_R(x); \mathbb{R}^N)$ and

$$
\|Xu\|_{L^p(B_R(x); \mathbb{R}^N)} \leq C \left( \|F\|_{L^p(B_{11R}(x); \mathbb{M}^{N\times q})} + \|Xu\|_{L^2(B_{11R}(x); \mathbb{R}^N)} \right).
$$

Proof of Theorem 2.12 from Theorem 2.13. For fixed domains $\Omega' \subset \Omega'' \subset \Omega$, pick $R_0$ such that $B_{12R_0}(x) \subset \Omega''$ for any $x \in \Omega'$. For this $R_0$ and a fixed $p \in (2, \infty)$, let $\delta$ be as in Theorem 2.13. Since the $a_{ij}^\beta$’s belong to $VMO_{loc}(\Omega)$, there exists $R \leq R_0$, $R$ depending on $\Omega$, $\Omega'$, $R_0$, $\delta$, such that the $a_{ij}^\beta$’s are $(\delta, 8R)$-vanishing in $B_R(x)$. Therefore by Theorem 2.13, (2.11) holds for any such $x$ and $R$. Next, we apply Caccioppoli’s inequality (2.9), getting

$$
\|Xu\|_{L^2(B_{11R}(x); \mathbb{R}^N)} \leq C \left( \frac{1}{R} \|u\|_{L^2(B_{12R}(x); \mathbb{R}^N)} + \|F\|_{L^2(B_{12R}(x); \mathbb{M}^{N\times q})} \right).
$$
which inserted in (2.11) gives
\[ \|Xu\|_{L^p(B_R(\mathbb{R}^N); \mathbb{R}^N)} \leq c(R) \left\{ \|F\|_{L^p(B_{12R}(\mathbb{R}^N); \mathbb{R}^{N \times q})} + \|u\|_{L^2(B_{12R}(\mathbb{R}^N))} \right\}. \tag{2.12} \]

On the other hand, by Poincaré’s inequality (2.2) we have
\[ \|u^\alpha\|_{L^p(B_R(\mathbb{R}^N))} \leq \left\{ \|u^\alpha - u^\alpha_{B_R(\mathbb{R}^N)}\|_{L^p(B_R(\mathbb{R}^N))} + |B_R(\mathbb{R}^N)|^{1/p} \right\} \leq cR \left\{ \|Xu^\alpha\|_{L^p(B_R(\mathbb{R}^N))} + \|u^\alpha\|_{L^2(B_R(\mathbb{R}^N))} \right\}^{1/p-1/2}; \]

hence
\[ \|u\|_{L^p(B_R(\mathbb{R}^N); \mathbb{R}^N)} \leq c(R, p) \left\{ \|Xu\|_{L^p(B_R(\mathbb{R}^N); \mathbb{R}^N)} + \|u\|_{L^2(B_R(\mathbb{R}^N))} \right\}, \]

which together with (2.12) gives
\[ \|u\|_{H^1 W^{1,p}(B_R(\mathbb{R}^N); \mathbb{R}^N)} \leq c \left\{ \|F\|_{L^p(B_{12R}(\mathbb{R}^N); \mathbb{R}^{N \times q})} + \|u\|_{L^2(B_{12R}(\mathbb{R}^N))} \right\}. \]

A covering argument then gives (2.10). □

It is worthwhile to point out that, as we will see from the proof of Theorem 2.13 in Section 5, the following bound, stronger than (2.11), is actually established:
\[ \|\mathcal{M}_{B_{11R}}(|Xu|^2)\|_{L^{p/2}(B_R(\mathbb{R}^N))} \leq c \left\{ \|F\|_{L^p(B_{12R}(\mathbb{R}^N); \mathbb{R}^{N \times q})} + \|u\|_{L^2(B_{12R}(\mathbb{R}^N))} \right\}, \tag{2.13} \]

where \(\mathcal{M}\) is the Hardy–Littlewood maximal function (see Section 4).

Remark 2.14. Note that what allows to exploit the VMO assumption on the coefficients is the fact that the number \(\delta\) in Theorem 2.13 depends on \(R_0\) but not on \(R \leq R_0\), which allows shrinking \(R\) without changing \(\delta\), to get the \((\delta, R)\)-vanishing condition satisfied. Under this regard, our result is very different from those proved for instance in [9,8] where the parameter \(\delta\) possibly depends on \(R\), which makes the \((\delta, R)\)-vanishing assumption hard to check.

Dependence of constants. Throughout this paper, the letter \(c\) denotes a constant which may vary from line to line. The parameters which the constants depend on are declared in the statements or in the proofs of the theorems. When we write that \(c\) is an “absolute constant” we mean that it may depend on \(G\) and \(N\).

3. Approximation by solutions of systems with constant coefficients

Notation 3.1. In order to simplify notation, henceforth we will systematically write the norms and spaces of vector valued functions as
\[ H^1 W^{1,p}(B); \mathbb{R}^N, \|u\|_{H^1 W^{1,p}(B; \mathbb{R}^N)}, \|F\|_{L^p(B; \mathbb{R}^{N \times q})} \]

instead of
\[ H^1 W^{1,p}(B; \mathbb{R}^N), \|u\|_{H^1 W^{1,p}(B; \mathbb{R}^N)}, \|F\|_{L^p(B; \mathbb{R}^{N \times q})}, \]

and so on.

In this section we will prove a couple of theorems asserting that a solution to a system (1.1) with small datum \(F\) and coefficients with small oscillation, can be suitably approximated by a solution to a system with constant coefficients and zero datum. This approximation is one of the tools which will be used in the proof of Theorem 2.13.

Theorem 3.2. Under Assumption (H) (see Section 2.2), for any \(\varepsilon > 0\), \(R_0 > 0\) there is a small \(\delta = \delta(\varepsilon, R_0, \mu) > 0\) such that for any \(R \geq R_0\), if \(u\) is a weak solution to the system (1.1) in \(B_{4R} \subseteq \Omega\)

\[ \frac{1}{|B_{4R}|} \int_{B_{4R}} |Xu|^2 \, dx \leq 1, \quad \frac{1}{|B_{4R}|} \int_{B_{4R}} |F|^2 + \left| a^\alpha_{\nu^\beta} - \left( a^\nu_{\alpha^\beta} \right)_{B_{4R}} \right|^2 \, dx \leq \delta^2, \tag{3.1} \]

then there exists a weak solution \(v\) to the following homogeneous system with constant coefficients:

\[ X_i \left( a^\nu_{\alpha^\beta} \right)_{B_{4R}} X_j v^\beta(x) = 0 \quad \text{in } B_{4R} \tag{3.2} \]

such that
\[ \frac{1}{R^2} \frac{1}{|B_{4R}|} \int_{B_{4R}} |u - v|^2 \, dx \leq \varepsilon^2. \]
Proof. Let us first prove the result for a fixed $R$ (and $\delta$ possibly depending on $R$), then we will show how to remove the dependence on $R$.

By contradiction, if the result does not hold, then there exist a constant $\varepsilon_0 > 0$, and sequences $\{a_{ijk}^{\alpha} \}_{k=1}^{\infty}$ satisfying (2.7), $\{u_k\}_{k=1}^{\infty}$, $\{F_k\}_{k=1}^{\infty}$ such that $u_k$ is a weak solution to the system

$$X_0\left(a_{ijk}^{\alpha} X_j^{\beta} (x)\right) = X_j f^k (x)$$

in $B_{3R}$ with

$$\frac{1}{|B_{3R}|} \int_{B_{3R}} |Xu_k|^2 \, dx \leq 1, \quad \frac{1}{|B_{4}|} \int_{B_{4}} \left( |F_k|^2 + |a_{ijk}^{\alpha} - (a_{ijk}^{\alpha})_{B_{4R}}|^2 \right) \, dx \leq \frac{1}{k^2},$$

but

$$\frac{1}{R^2} \frac{1}{|B_{3R}|} \int_{B_{3R}} |u_k - v_k|^2 \, dx > \varepsilon_0^2$$

for any weak solution $v_k$ of

$$X_0\left(\left(a_{ijk}^{\alpha} \right)_{B_{3R}} X_j^{\beta} (x)\right) = 0 \quad \text{in } B_{3R}.$$ \hspace{1cm} (3.3)

From (3.4) and Poincaré’s inequality (2.2), we know that $\{u_k - (u_k)_{B_{3R}}\}_{k=1}^{\infty}$ is bounded in $H^{1,2}(B_{3R})$, then Rellich’s lemma allows us to find a subsequence of $\{u_k - (u_k)_{B_{3R}}\}$, still denoted by $\{u_k - (u_k)_{B_{3R}}\}$, such that

$$\frac{1}{R^2} \frac{1}{|B_{3R}|} \int_{B_{3R}} |u_k - (u_k)_{B_{3R}} - u_0|^2 \, dx \to 0,$$

$$Xu_k \to Xu_0 \text{ weakly in } L^2,$$ \hspace{1cm} (3.5)

as $k \to \infty$, for some $u_0 \in H^{1,2}(B_{3R})$. Since $\left\{a_{ijk}^{\alpha} \right\}_{k=1}^{\infty}$ is bounded in $\mathbb{R}$, it allows a subsequence, still denoted by $\left\{a_{ijk}^{\alpha} \right\}_{k=1}^{\infty}$, such that

$$\left|\left(\left(a_{ijk}^{\alpha} \right)_{B_{3R}} - \tilde{a}_{ijk}^{\alpha} \right)\right| \to 0, \quad \text{as } k \to \infty,$$ \hspace{1cm} (3.6)

for some constants $\tilde{a}_{ijk}^{\alpha}$. By (3.4), it follows

$$a_{ijk}^{\alpha} \to \tilde{a}_{ijk}^{\alpha} \text{ in } L^2(B_{3R}), \quad \text{as } k \to \infty.$$ \hspace{1cm} (3.7)

Next, we show that $u_0$ is a weak solution of

$$X_0\left(\tilde{a}_{ijk}^{\alpha} X_j^{\beta} (x)\right) = 0 \quad \text{in } B_{3R}.$$ \hspace{1cm} (3.8)

We start from

$$\int_{B_{3R}} a_{ijk}^{\alpha} (x) X_j u_k^{\beta} \psi^{\alpha} \, dx = \int_{B_{3R}} f^k X_0 \psi^{\alpha} \, dx$$

with $\psi^{\alpha} \in C_0^\infty(\Omega)$, and take the limit for $k \to \infty$. By (3.4),

$$\int_{B_{3R}} f^k X_0 \psi^{\alpha} \, dx \to 0.$$ \hspace{1cm} (3.9)

Moreover,

$$\int_{B_{3R}} a_{ijk}^{\alpha} (x) X_j u_k^{\beta} \psi^{\alpha} \, dx = \int_{B_{3R}} \left[ a_{ijk}^{\alpha} (x) - \tilde{a}_{ijk}^{\alpha} \right] X_j u_k^{\beta} \psi^{\alpha} \, dx + \int_{B_{3R}} \tilde{a}_{ijk}^{\alpha} X_j u_k^{\beta} \psi^{\alpha} \, dx \equiv A_k + B_k.$$ \hspace{1cm} (3.10)

Now,

$$|A_k| \leq c \left\| a_{ijk}^{\alpha} (x) - \tilde{a}_{ijk}^{\alpha} \right\|_{L^2(B_{3R})} \left\| X_j u_k^{\beta} \right\|_{L^2(B_{3R})} \to 0,$$ \hspace{1cm} (3.11)
because \( a_{ij}^\alpha (x) \to a_{ij}^\alpha \) in \( L^2 \) and \( \{ \chi u_0^\alpha \} \) is bounded in \( L^2 \). Finally, since \( Xu_k \to Xu_0 \) weakly in \( L^2 \),
\[
B_k \to \int_{B_{4R}} a_{ij}^{\alpha \beta} X_i u_0^\alpha X_j \phi^\alpha dx;
\]
hence
\[
\int_{B_{4R}} a_{ij}^{\alpha \beta} X_i u_0^\alpha X_j \phi^\alpha dx = 0 \quad \text{for any } \phi^\alpha \in C_0^\infty (B_{4R}).
\]
By density, this holds for any \( \phi^\alpha \in HW_0^{1,2} (B_{4R}) \), so \( u_0 \) is a weak solution to (3.10).

Now, let \( v_k \) be the unique solution to the Dirichlet problem
\[
\begin{align*}
\left\{ X_i \left( a_{ij}^{\alpha \beta} \right)_B X_i v_k \right\} &= 0 \text{ in } B_{4R} \\
v_k - u_0 &\in HW_0^{1,2} (B_{4R})
\end{align*}
\]
(see Remark 2.9). By (2.7) and using \( v_k - u_0 \) as a test function in the definition of the solution to (3.12) we have
\[
\mu \int_{B_{4R}} \{ X v_k - Xu_0 \}^2 dx \leq \int_{B_{4R}} \left( a_{ij}^{\alpha \beta} \right)_B \{ X v_k^\alpha - X u_0^\alpha \} \{ X v_k^\beta - X u_0^\beta \} dx
\]
\[
= - \int_{B_{4R}} \left( a_{ij}^{\alpha \beta} \right)_B X_i u_0^\beta \{ X v_k^\alpha - X u_0^\alpha \} dx,
\]
so \( u_0 \) is a weak solution to (3.10)
\[
= \int_{B_{4R}} \left( \tilde{a}_{ij}^{\alpha \beta} \right)_B \{ X v_k^\alpha - X u_0^\alpha \} dx
\]
\[
\leq c \left( N \max_{i,j,\alpha,\beta} \left| \left( \tilde{a}_{ij}^{\alpha \beta} \right)_B - \left( a_{ij}^{\alpha \beta} \right)_B \right| \right) \left( \int_{B_{4R}} |X u_0|^2 dx \right)^{1/2} \cdot \left( \int_{B_{4R}} |X v_k - Xu_0|^2 dx \right)^{1/2},
\]
which implies
\[
\mu \left( \int_{B_{4R}} |X v_k - Xu_0|^2 dx \right)^{1/2} \leq c \max_{i,j,\alpha,\beta} \left| \left( \tilde{a}_{ij}^{\alpha \beta} \right)_B - \left( a_{ij}^{\alpha \beta} \right)_B \right| \left( \int_{B_{4R}} |X u_0|^2 dx \right)^{1/2}.
\]
Inequalities (3.9) and (3.13) imply
\[
|X v_k - Xu_0|_{L^2 (B_{4R})} \to 0 \quad \text{as } k \to 0.
\]
This convergence, the fact that \( v_k \to u_0 \in HW_0^{1,2} (B_{4R}) \) and (2.3) imply
\[
|v_k - u_0|_{L^2 (B_{4R})} \to 0 \quad \text{as } k \to 0.
\]
By (3.7) and (3.14) we can write
\[
\| v_k - (u_k)_{B_{4R}} \|_{L^2 (B_{4R})} \leq \| u_0 - (u_k)_{B_{4R}} \|_{L^2 (B_{4R})} + \| v_k - u_0 \|_{L^2 (B_{4R})} \to 0.
\]
On the other hand, \( v_k + (u_k)_{B_{4R}} \) is still a weak solution to (3.6); hence (3.5) implies
\[
\frac{1}{R^2} \int_{B_{4R}} \left| v_k - (u_k)_{B_{4R}} \right|^2 dx > \epsilon_0^2,
\]
which contradicts (3.15). So we have proved the assertion, for some \( \delta \) possibly depending on \( \epsilon, R, \mu \).

Let us now fix a particular \( R_0 \), and let \( R \) be any number \( \leq R_0 \). Assume \( u \) is a weak solution to system (1.1) in \( B_{4R} \subseteq \Omega \) satisfying (3.1). Just to simplify notations, assume that the center of \( B_{4R} \) is the origin, and define
\[
\tilde{u} (x) = \frac{R_0}{R} u \left( R \frac{R_0}{R} x \right);
\]
\[
\tilde{a}_{ij}^{\alpha \beta} (x) = a_{ij}^{\alpha \beta} \left( R \frac{R_0}{R} x \right);
\]
\[
\tilde{f}_\alpha (x) = f_\alpha \left( R \frac{R_0}{R} x \right).
\]
Then, one can check that the function $\tilde{u}$ solves the system
\[
X_i \left( \tilde{a}_{ij}^\alpha (x) X_j \tilde{u}^\beta \right) = X_i \tilde{f}_j^\alpha \quad \text{in } B_{4R_0}.
\]
To see this, for any $\phi \in C_0^\infty (B_{4R})$, let $\tilde{\phi} (x) = \frac{R_0}{R} \phi \left( \frac{R}{R_0} x \right)$; then $\tilde{\phi} \in C_0^\infty (B_{4R_0})$ and
\[
\int_{B_{4R_0}} \tilde{a}_{ij}^\alpha (x) X_j \tilde{u}^\beta (x) X_i \tilde{\phi}^\alpha (x) \, dx = \int_{B_{4R_0}} a_{ij}^\alpha \left( \frac{R}{R_0} x \right) \left( X_j u^\beta \right) \left( \frac{R}{R_0} x \right) \left( X_i \phi^\alpha \right) \left( \frac{R}{R_0} x \right) \, dx \nonumber
\]
\[
= \left( \frac{R_0}{R} \right)^Q \int_{B_{4R_0}} a_{ij}^\alpha (y) \left( X_j u^\beta \right) (y) \left( X_i \phi^\alpha \right) (y) \, dy \nonumber
\]
\[
= \left( \frac{R_0}{R} \right)^Q \int_{B_{4R_0}} f_i^\alpha (y) \left( X_i \phi^\alpha \right) (y) \, dy \nonumber
\]
\[
= \int_{B_{4R_0}} f_i^\alpha \left( \frac{R}{R_0} x \right) \left( X_i \phi^\alpha \right) \left( \frac{R}{R_0} x \right) \, dx \nonumber
\]
\[
= \int_{B_{4R_0}} \tilde{f}_i^\alpha (x) X_i \tilde{\phi}^\alpha (x) \, dx.
\]
Also, note that the $\tilde{a}_{ij}^\alpha$'s satisfy condition (2.7) with the same $\mu$. Let $\delta = \delta (\varepsilon, R_0, \mu)$ be the number found in the first part of the proof, and assume that $u, F, a_{ij}^{\alpha \beta}$ satisfy (3.1) on $B_{4R}$ for this $\delta$; then $\tilde{u}, \tilde{F}, \tilde{a}_{ij}^{\alpha \beta}$ satisfy (3.1) on $B_{4R_0}$ for the same $\delta$:
\[
\frac{1}{|B_{4R_0}|} \int_{B_{4R_0}} |\tilde{u} (x)|^2 \, dx = \frac{1}{|B_{4R_0}|} \int_{B_{4R_0}} |(Xu) \left( \frac{R}{R_0} x \right)|^2 \, dx \nonumber
\]
\[
= \frac{1}{|B_{4R_0}|} \left( \frac{R_0}{R} \right)^Q \int_{B_{4R_0}} |(Xu) (y)|^2 \, dy = \frac{1}{|B_{4R}|} \int_{B_{4R}} |(Xu) (y)|^2 \, dy \leq 1; \nonumber
\]
\[
\frac{1}{|B_{4R_0}|} \int_{B_{4R_0}} \left( |\tilde{F}|^2 + |\tilde{a}_{ij}^{\alpha \beta} - (\tilde{a}_{ij}^{\alpha \beta})_{B_{4R_0}} |^2 \right) \, dx = \frac{1}{|B_{4R_0}|} \int_{B_{4R_0}} \left( |F|^2 + |a_{ij}^{\alpha \beta} - (a_{ij}^{\alpha \beta})_{B_{4R}} |^2 \right) \, dx \leq \delta. \nonumber
\]
Hence, by the first part of the proof, there exists a weak solution $\tilde{v}$ to the following homogeneous system with constant coefficients:
\[
X_i \left( \tilde{a}_{ij}^\alpha \right) X_j \tilde{v}^\beta (x) = 0 \quad \text{in } B_{4R_0}
\]
such that
\[
\frac{1}{R^2} \frac{1}{|B_{4R_0}|} \int_{B_{4R_0}} |\tilde{u} - \tilde{v}|^2 \, dx \leq \varepsilon^2.
\]
Then, the function
\[
u (x) = \frac{R}{R_0} \tilde{v} \left( \frac{R_0}{R} x \right)
\]
satisfies
\[
X_i \left( \tilde{a}_{ij}^\alpha \right) X_j \nu^\beta (x) = 0 \quad \text{in } B_{4R}
\]
and
\[
\frac{1}{R^2} \frac{1}{|B_{4R}|} \int_{B_{4R}} |u (x) - \nu (x)|^2 \, dx = \frac{1}{R^2} \frac{1}{|B_{4R}|} \int_{B_{4R}} \left| \frac{R}{R_0} \tilde{u} \left( \frac{R_0}{R} x \right) - \frac{R}{R_0} \tilde{v} \left( \frac{R_0}{R} x \right) \right|^2 \, dx \nonumber
\]
\[
= \frac{1}{R^2} \frac{1}{|B_{4R}|} \int_{B_{4R}} \left| \tilde{u} \left( \frac{R_0}{R} x \right) - \tilde{v} \left( \frac{R_0}{R} x \right) \right|^2 \, dx \nonumber
\]
\[
= \frac{1}{R^2} \frac{1}{|B_{4R_0}|} \int_{B_{4R_0}} |\tilde{u} - \tilde{v}|^2 \, dx \leq \varepsilon^2.
\]
We have therefore proved that the assertion holds with $\delta$ depending on $R_0$ but independent of $R \leq R_0$. □
The following technical lemma is adapted from [11, Lemma 4.1, p.27].

**Lemma 3.3.** Let \( \psi(t) \) be a bounded nonnegative function defined on the interval \([T_0, T_1]\), where \( T_1 > T_0 \geq 0 \). Suppose that for any \( T_0 \leq t \leq s \leq T_1 \), \( \psi \) satisfies
\[
\psi(t) \leq \vartheta \psi(s) + \frac{A}{(s-t)\beta} + B,
\]
where \( \vartheta, A, B, \beta \) are nonnegative constants, and \( \vartheta < \frac{1}{3} \). Then
\[
\psi(\rho) \leq c_\beta \left[ \frac{A}{(R-\rho)\beta} + B \right], \quad \forall \rho, T_0 \leq \rho < R \leq T_1,
\]
where \( c_\beta \) only depends on \( \beta \).

We are going to enforce the previous theorem with the following.

**Theorem 3.4.** For any \( \varepsilon > 0 \), \( R_0 > 0 \), there is a small \( \delta = \delta (\varepsilon, R_0, \mu) > 0 \) such that for any \( R \leq R_0 \), if \( u \) is a weak solution of system (1.1) in \( B_{4R} \subseteq \Omega \) and (3.1) holds, then there exists a weak solution \( v \) to (3.2) such that
\[
\frac{1}{|B_{2R}|} \int_{B_{2R}} |Xu - Xu|^2 \, dx \leq \varepsilon^2.
\]

**Proof.** By Theorem 3.2, we know that for any \( \eta > 0 \), there exist a small \( \delta = \delta (\eta, R_0, \mu) > 0 \) and a weak solution \( v \) of (3.2) in \( B_{4R} \), such that
\[
\frac{1}{R^2} \frac{1}{|B_{4R}|} \int_{B_{4R}} |u - v|^2 \, dx \leq \eta^2,
\]
provided (3.1) holds.

Let us note that \( u - v \) is a weak solution to the system
\[
X_i \left( a^{ij}_{\alpha\beta} (x) X_j (u^\alpha - v^\alpha) (x) \right) = X_i \left( f^i_\alpha (x) - \left( a^{ij}_{\alpha\beta} (x) - \left( a^{ij}_{\alpha\beta} \right)_{B_{4R}} \right) X_j v^\beta \right)
\]
in \( B_{4R} \). For any \( 2R \leq s < t \leq 3R \), we choose a cutoff function \( \psi (x) \) which satisfies
\[
0 < \psi (x) \leq 1 \quad \text{in} \ B_{3R}, \quad \psi (x) \equiv 1 \quad \text{in} \ B_{s}, \quad \psi (x) \equiv 0 \quad \text{in} \ B_{3R} \setminus B_{t}
\]
and
\[
|X\psi (x)| \leq \frac{c}{t-s} \quad \text{in} \ B_{4R}.
\]

Taking \( (u - v) \psi \) as a test function, it follows by (3.17) that
\[
\mu \int_{B_t} |X (u - v)|^2 \, dx \leq \int_{B_t} \psi (x) a^{ij}_{\alpha\beta} (x) X_j (u^\alpha - v^\alpha) X_i (u^\alpha - v^\alpha) \, dx
\]
\[
= \int_{B_t} \left( f^i_\alpha (x) - \left( a^{ij}_{\alpha\beta} (x) - \left( a^{ij}_{\alpha\beta} \right)_{B_{4R}} \right) X_j v^\beta \right) X_i (u^\alpha - v^\alpha) \psi \, dx
\]
\[
- \int_{B_t} a^{ij}_{\alpha\beta} (x) (u^\alpha - v^\alpha) X_j (u^\beta - v^\beta) X_i \psi \, dx.
\]

By the properties of \( \psi \), Young’s inequality and (2.7),
\[
\int_{B_t} |Xu - Xu|^2 \, dx \leq c \int_{B_t} \left( |F| + \max_{i,j,\alpha,\beta} \left| a^{ij}_{\alpha\beta} (x) - \left( a^{ij}_{\alpha\beta} \right)_{B_{4R}} \right| |Xu| \right)^2 \, dx
\]
\[
+ \frac{1}{4} \int_{B_t} |Xu - Xu|^2 \, dx + \frac{c}{(t-s)^2} \int_{B_t} |u - v|^2 \, dx
\]
\[
\leq c \int_{B_t} |F|^2 \, dx + \sup_{B_{3R}} |Xu|^2 \cdot \max_{i,j,\alpha,\beta} \int_{B_{4R}} \left| a^{ij}_{\alpha\beta} (x) - \left( a^{ij}_{\alpha\beta} \right)_{B_{4R}} \right|^2 \, dx
\]
\[
+ \frac{c}{(t-s)^2} \int_{B_{4R}} |u - v|^2 \, dx + \frac{1}{4} \int_{B_t} |Xu - Xu|^2 \, dx.
\]
Setting
\[ \psi (s) = \int_{B_1} |Xu - Xv|^2 \, dx, \]
\[ B = c \int_{B_{2R}} |F|^2 \, dx + \sup_{B_{2R}} |Xv|^2 \cdot \max_{i,j,\alpha,\beta} \int_{B_{2R}} \left| a_{ij}^{\alpha \beta} (x) - \left( a_{ij}^{\alpha \beta} (x) \right)_{B_{2R}} \right|^2 \, dx, \]
\[ A = \int_{B_{2R}} |u - v|^2 \, dx, \quad \beta = 2, \]
by Lemma 3.3 we deduce
\[ \int_{B_{2R}} |Xu - Xv|^2 \, dx \leq \frac{c}{R^2} \int_{B_{2R}} |u - v|^2 \, dx + c \int_{B_{2R}} |F|^2 \, dx \]
\[ + c \sup_{B_{2R}} |Xv|^2 \cdot \max_{i,j,\alpha,\beta} \int_{B_{2R}} \left| a_{ij}^{\alpha \beta} (x) - \left( a_{ij}^{\alpha \beta} (x) \right)_{B_{2R}} \right|^2 \, dx. \] (3.18)
By Theorem 2.10, since \( v - u_{B_{2R}} \) is still a solution to the system (3.2) in \( B_{2R} \) we can write
\[ \sup_{B_{2R}} |Xv| \leq \frac{c}{R} |B_R|^{-1/2} \left\| v - u_{B_{2R}} \right\|_{L^2(B_{2R})} \]
\[ \leq \frac{c}{R} |B_R|^{-1/2} \left( \left\| u - v \right\|_{L^2(B_{2R})} + \left\| u - u_{B_{2R}} \right\|_{L^2(B_{2R})} \right) \]
by (3.16), (2.2) and assumption (3.1) on \( u \)
\[ \leq c \eta + c |B_R|^{-1/2} \left\| Xu \right\|_{L^2(B_{2R})} \leq c (\eta + 1) \leq N_0, \] (3.19)
for some absolute constant \( N_0 \) when \( \eta \) is, say, any number \( \leq 1 \).
By (3.18) and (3.19) we have
\[ \frac{1}{|B_{2R}|} \int_{B_{2R}} |Xu - Xv|^2 \, dx \leq \frac{c}{|B_{2R}|} \int_{B_{2R}} |F|^2 \, dx + \frac{cN_0}{|B_{2R}|} \max_{i,j,\alpha,\beta} \int_{B_{2R}} \left| a_{ij}^{\alpha \beta} (x) - \left( a_{ij}^{\alpha \beta} (x) \right)_{B_{2R}} \right|^2 \, dx \]
\[ + \frac{c}{R^2} \frac{1}{|B_{2R}|} \int_{B_{2R}} |u - v|^2 \, dx, \]
by (3.16) and (3.1)
\[ \leq \frac{c}{|B_{2R}|} \int_{B_{2R}} \left( |F|^2 + \max_{i,j,\alpha,\beta} \left| a_{ij}^{\alpha \beta} (x) - \left( a_{ij}^{\alpha \beta} (x) \right)_{B_{2R}} \right|^2 \right) \, dx + c \eta^2 \]
\[ \leq c (\delta^2 + \eta^2) < \varepsilon^2, \]
for a suitable choice of \( \eta \), and after possibly diminishing \( \delta \). This ends the proof. \( \square \)

4. Estimates on the maximal function of \( |Xu|^2 \)

**Definition 4.1.** Let \( B_R \subset \Omega \). For every \( f \in L^1 (B_R) \), define the Hardy–Littlewood maximal function of \( f \) by
\[ \mathcal{M}_{B_R} (f) (x) = \sup_{r > 0} \frac{1}{|B_r (x) \cap B_R|} \int_{B_r (x) \cap B_R} |f (y)| \, dy. \]

Since \( (B_R, d_\chi, dx) \) is a space of homogeneous type (see Remark 2.7), by [16, Theorem 2.1 p. 71] the following holds.

**Lemma 4.2.** Let \( f \in L^1 (B_R) \), then
(i) \( \mathcal{M}_{B_R} (f) (x) \) is finite almost everywhere in \( B_R \);
(ii) for every \( \alpha > 0 \),
\[ \left\{ x \in B_R : \mathcal{M}_{B_R} (f) (x) > \alpha \right\} \leq \frac{c_1}{\alpha} \int_{B_R} |f (y)| \, dy; \]
(iii) if \( f \in L^p (B_R) \) with \( 1 < p < \infty \), then \( \mathcal{M}_{B_R} (f) \in L^p (B_R) \) and
\[
\| \mathcal{M}_{B_R} (f) \|_{L^p (B_R)} \leq c_p \| f \|_{L^p (B_R)},
\]
where the constants \( c_p \) only depend on \( p \) and \( G \) (but are independent of \( B_R \)).

The last statement about the dependence of the constants requires some explanation. In any space of homogeneous type these constants depend on the two constants of the space, namely the one appearing in the “quasitriangle inequality” (2.4) and the doubling constant appearing in (2.5). In our case the first constant is 1 (since \( d_x \) is a distance) and the second is independent of \( R \), by Remark 2.7. Hence \( c_p \) is independent of \( R \).

**Theorem 4.3.** There exists an absolute constant \( N_1 \) such that for any \( \varepsilon > 0 \), \( R_0 > 0 \), there is a small \( \delta = \delta (\varepsilon, R_0, \mu) > 0 \) such that for any \( R \leq R_0 / 2, z \in B_R (\bar{x}) \subset B_{11R} (\bar{x}) \in \Omega \) and \( 0 < r \leq 2R \), if \( u \) is a weak solution of (1.1) in \( B_{11R} (\bar{x}) \) with
\[
\mathcal{B}_r (z) \cap \left\{ x \in B_R (\bar{x}) : \mathcal{M}_{B_{11R}(\bar{x})} (|Xu|^2) (x) \leq 1 \right\} \cap \left\{ x \in B_R (\bar{x}) : \mathcal{M}_{B_{11R}(\bar{x})} (|F|^2) (x) \leq \delta^2 \right\} \neq \emptyset
\]
and the coefficients \( a_{\alpha \beta} (x) \) are \((\delta, 4r)\)-vanishing in \( B_R (\bar{x}) \), then
\[
\mathcal{B}_r (z) \cap \left\{ x \in B_R (\bar{x}) : \mathcal{M}_{B_{11R}(\bar{x})} (|Xu|^2) (x) > N_1^2 \right\} < \varepsilon \mathcal{B}_r (z).
\]

**Proof.** Fix \( \varepsilon, R_0 > 0 \); the number \( \delta \) will be chosen later. By (4.1), there exists a point \( x_0 \in \mathcal{B}_r (z) \), such that for any \( \rho > 0 \),
\[
\frac{1}{|B_\rho (x_0) \cap B_{11R} (\bar{x})|} \int_{B_\rho (x_0) \cap B_{11R} (\bar{x})} |Xu|^2 \, dx \leq 1, \tag{4.3}
\]
\[
\frac{1}{|B_\rho (x_0) \cap B_{11R} (\bar{x})|} \int_{B_\rho (x_0) \cap B_{11R} (\bar{x})} |F|^2 \, dx \leq \delta^2. \tag{4.4}
\]
Since \( z, x_0 \in B_R (\bar{x}) \) and \( r \leq 2R \), we have the inclusions: \( \mathcal{B}_{4r} (z) \subset B_{5r} (x_0) \subset B_{11R} (\bar{x}) \) and \( B_{5r} (x_0) \subset B_{6r} (z) \). Then by (4.4) with \( \rho = 5r \) we have that
\[
\frac{1}{|B_{4r} (z)|} \int_{B_{4r} (z)} |F|^2 \, dx \leq \frac{|B_{5r} (z)|}{|B_{4r} (z)|} \frac{1}{|B_{5r} (x_0)|} \int_{B_{5r} (x_0)} |F|^2 \, dx \leq \left( \frac{6}{4} \right)^Q \delta^2. \tag{4.5}
\]
Similarly, by (4.3) we find
\[
\frac{1}{|B_{4r} (z)|} \int_{B_{4r} (z)} |Xu|^2 \, dx \leq \left( \frac{6}{4} \right)^Q. \tag{4.6}
\]
By (4.5), (4.6) and the assumption on \( a_{\alpha \beta} (x) \), we can apply Theorem 3.4 (with \( u \) replaced by \( \left( \frac{\varepsilon}{\delta} \right)^Q u \) and \( F \) replaced by \( \left( \frac{\varepsilon}{\delta} \right)^Q F \) on the ball \( B_{4r} (z) \) (recall that \( r \leq R_0 \)) and obtain that for any \( \eta > 0 \), there exists a small \( \delta = \delta (\eta, R_0, \mu) \) and a weak solution \( v \) to
\[
X_i \left( \left( \frac{a_{\alpha \beta}}{B_{4r} (z)} \right)^{\left( \frac{\varepsilon}{\delta} \right)^Q} X^j v = 0 \right) \text{ in } B_{4r} (z)
\]
such that
\[
\frac{1}{|B_{2r} (z)|} \int_{B_{2r} (z)} |X (u - v)|^2 \, dx \leq \eta^2. \tag{4.7}
\]
Also, recall the interior \( HW^{1, \infty} \) regularity of \( v \) (3.19):
\[
\|Xu\|_{L^2 (B_{4r} (z))}^2 < N_0^2. \tag{4.8}
\]
Now, pick
\[
N_1^2 = \max \left\{ \frac{5^Q}{c_d}, 4N_0^2 \right\}. \tag{4.9}
\]
Then we claim that
\[
\left\{ x \in B_R (\bar{x}) : \mathcal{M}_{B_{11R}(\bar{x})} (|Xu|^2) (x) > N_1^2 \right\} \cap B_r (z)
\]
\[
\subset \left\{ x \in B_R (\bar{x}) : \mathcal{M}_{B_{2r}(z)} (|X (u - v)|^2) (x) > N_0^2 \right\} \cap B_r (z). \tag{4.10}
\]
To see this, suppose
\[ x_1 \in \left\{ x \in B_R(\bar{x}) \cap B_r(z) : \mathcal{M}_{B_{2r}(z)} \left( |X(u-v)|^2 \right)(x) \leq N_0^2 \right\}. \tag{4.11} \]
When \( \rho \leq r \), it follows that \( B_{\rho}(x_1) \subset B_{2r}(z) \subset B_{2\rho}(\bar{x}) \), then (4.11) and (4.8) imply
\[
\frac{1}{|B_{\rho}(x_1) \cap B_{11R}(\bar{x})|} \int_{B_{\rho}(x_1) \cap B_{11R}(\bar{x})} |Xu|^2 \, dx = \frac{1}{|B_{\rho}(x_1)|} \int_{B_{\rho}(x_1)} |Xu|^2 \, dx
\leq \frac{2}{|B_{\rho}(x_1)|} \int_{B_{\rho}(x_1)} (|X(u-v)|^2 + |Xv|^2) \, dx \leq 4N_0^2 \leq N_1^2. \tag{4.12}
\]
When \( \rho > r \), since \( x_1, x_0 \in B_r(z) \) we have \( d(x_1, x_0) < 2r < 2\rho \); it follows that \( B_{\rho}(x_1) \subset B_{3\rho}(x_0) \subset B_{5\rho}(x_1) \). Then by Remark 2.7 and (4.3) we have
\[
\frac{1}{|B_{\rho}(x_1) \cap B_{11R}(\bar{x})|} \int_{B_{\rho}(x_1) \cap B_{11R}(\bar{x})} |Xu|^2 \, dx \leq \frac{1}{cd |B_{\rho}(x_1)|} \int_{B_{3\rho}(x_0) \cap B_{11R}(\bar{x})} |Xu|^2 \, dx
= \frac{5^Q}{cd |B_{3\rho}(x_0)|} \int_{B_{3\rho}(x_0) \cap B_{11R}(\bar{x})} |Xu|^2 \, dx
\leq \frac{5^Q}{cd |B_{3\rho}(x_0)|} \leq N_1^2 \tag{4.13}.
\]
By (4.12) and (4.13), we have
\[ x_1 \in \left\{ x \in B_R(\bar{x}) : \mathcal{M}_{B_{11R}(\bar{x})} \left( |Xu|^2 \right) \leq N_1^2 \right\} \cap B_r(z). \tag{4.14} \]
Thus, inclusion (4.10) follows from the fact that (4.11) implies (4.14).

By (4.10), Lemma 4.2 (ii) and (4.7), we have
\[
\left| \left\{ x \in B_R(\bar{x}) : \mathcal{M}_{B_{11R}(\bar{x})} \left( |Xu|^2 \right)(x) > N_1^2 \right\} \cap B_r(z) \right|
\leq \frac{c}{N_0^2} \int_{B_{2r}(z)} |X(u-v)|^2 \, dx
\leq c\eta^2 |B_{2r}(z)| = c2^Q \eta^2 |B_r(z)|
= \varepsilon^2 |B_r(z)|.
\]
For a fixed \( \varepsilon \), we have finally chosen \( \eta \) so that \( c2^Q \eta^2 = \varepsilon^2 \) and picked the corresponding \( \delta \) depending on \( R_0, \mu \) and \( \eta \), that is on \( R_0, \mu, \varepsilon \). This finishes our proof. \( \square \)

**Corollary 4.4.** For any \( \varepsilon > 0, R_0 > 0 \), there is a small \( \delta = \delta (\varepsilon, R_0, \mu) > 0 \) such that for any \( R \leq R_0/2, z \in B_R(\bar{x}) \), \( 0 < r \leq 2R \), if \( u \) is a weak solution of (1.1) in \( B_{11R}(\bar{x}) \subseteq \Omega \), the coefficients \( a_{\mu, \lambda}^{\mu, \lambda} (x) \) are \( (\delta, 4r) \)-vanishing in \( B_R(\bar{x}) \) and
\[
\left| \left\{ x \in B_R(\bar{x}) : \mathcal{M}_{B_{11R}(\bar{x})} \left( |Xu|^2 \right)(x) > N_1^2 \right\} \cap B_r(z) \right| \geq \varepsilon |B_r(z)|,
\]
then
\[ B_r(z) \cap B_R(\bar{x}) \subset \left\{ x \in B_R(\bar{x}) : \mathcal{M}_{B_{11R}(\bar{x})} \left( |Xu|^2 \right)(x) > 1 \right\} \cup \left\{ x \in B_R(\bar{x}) : \mathcal{M}_{B_{11R}(\bar{x})} \left( \|u\|^2 \right)(x) > \delta^2 \right\}. \]

### 5. \( L^p \) estimate on \( |Xu| \)

In this section we exploit the local estimates on the maximal function of \( |Xu|^2 \) proved in the previous section in order to prove the desired \( L^p \) bound. The starting point is the following useful lemma about the estimate of the \( L^p \) norm of a function by means of its distribution function.

**Lemma 5.1** (See [10, p. 62]). Let \( \theta > 0, m > 1 \) be constants, \( p \in (1, \infty) \). Then there exists \( c > 0 \) such that for any nonnegative and measurable function \( f \) in \( \Omega \),
\[ f \in L^p (\Omega) \quad \text{if and only if} \quad S = \sum_{l \geq 1} m^p \left| \left\{ x \in \Omega : f(x) > \theta m^l \right\} \right| < \infty \]
and
\[ \frac{1}{c} S \leq \|f\|_{L^p(\Omega)} \leq c (|\Omega| + S). \]

**Lemma 5.2** (Vitali). Let \( F \) be a family of \( d_X \)-balls in \( \mathbb{R}^n \) with bounded radii. There exists a finite or countable sequence \( \{B_i\} \subset F \) of mutually disjoint balls such that
\[ \bigcup_{B \in F} B \subset \bigcup_i 5B_i, \]
where \( 5B \) is the ball with the same center as \( B \) and radius five times big.

The proof is identical to that of the Euclidean case, with the Euclidean distance replaced by \( d_X \) here.

**Lemma 5.3.** Let \( 0 < \varepsilon < 1, C \) and \( D \) be two measurable sets satisfying \( C \subset D \subset B_R(\bar{x}) \subset \Omega, |C| < \varepsilon |B_R(\bar{x})| \) and the following property:
\[ \forall \bar{x} \in B_R(\bar{x}), \quad \forall r \leq 2R, \quad |C \cap B_r(\bar{x})| \geq \varepsilon |B_r(\bar{x})| \implies B_r(\bar{x}) \cap B_R(\bar{x}) \subset D. \quad (5.1) \]
Then
\[ |C| \leq \varepsilon \frac{5^0}{c_d} |D|, \]
where \( c_d \) is the constants in (2.6).

**Proof.** For any \( \bar{x} \in C, C \subset B_R(\bar{x}) \subset B_{2R}(\bar{x}); \) hence
\[ |C \cap B_{2R}(\bar{x})| = |C| < \varepsilon |B_R(\bar{x})| < \varepsilon |B_{2R}(\bar{x})|. \]

On the other hand, by Lebesgue differentiation theorem, for a.e. \( \bar{x} \in C, \)
\[ \lim_{r \to 0} \frac{|C \cap B_r(\bar{x})|}{|B_r(\bar{x})|} = 1; \]
hence for a.e. \( \bar{x} \in C \) there is an \( r_\bar{x} \leq 2R \) such that for all \( r \in (r_\bar{x}, 2R) \) it holds
\[ |C \cap B_r(\bar{x})| \geq \varepsilon |B_r(\bar{x})| \quad \text{and} \quad |C \cap B_r(\bar{x})| < \varepsilon |B_r(\bar{x})|. \quad (5.2) \]

By Lemma 5.2, there are \( x_1, x_2, \ldots \in C, \) such that \( B_{5r_\bar{x}_1}(x_1), B_{5r_\bar{x}_2}(x_2), \ldots \) are mutually disjoint and satisfy
\[ \bigcup_k B_{5r_\bar{x}_k}(x_k) \cap B_R(\bar{x}) \supset C. \]

By (5.2) and (2.1), we know
\[ |C \cap B_{5r_\bar{x}_k}(x_k)| < \varepsilon |B_{5r_\bar{x}_k}(x_k)| = \varepsilon 5^0 |B_{r_\bar{x}_k}(x_k)|. \]

Also,
\[ |C| = \left| \bigcup_k B_{5r_\bar{x}_k}(x_k) \cap C \right| \leq \sum_k |B_{5r_\bar{x}_k}(x_k) \cap C| \]
\[ \leq \varepsilon 5^0 \sum_k |B_{r_\bar{x}_k}(x_k)| \]
\[ \leq \varepsilon \frac{5^0}{c_d} \sum_k |B_{r_\bar{x}_k}(x_k) \cap B_R(\bar{x})|, \]

where the last inequality follows since \( B_R(\bar{x}) \) is \( d_X \)-regular (see Remark 2.7). Moreover since the \( B_{r_\bar{x}_k}(x_k) \) are mutually disjoint the last quantity equals
\[ = \varepsilon \frac{5^0}{c_d} \left| \bigcup_k \left( B_{r_\bar{x}_k}(x_k) \cap B_R(\bar{x}) \right) \right| \leq \varepsilon \frac{5^0}{c_d} |D|, \]
since, by assumption (5.1), \( B_{r_\bar{x}_k}(x_k) \cap B_R(\bar{x}) \subset D. \) This completes the proof. \( \square \)
**Theorem 5.4.** For any $\varepsilon > 0$, $R_0 > 0$ there is a small $\delta = \delta (\varepsilon, R_0, \mu) > 0$ such that for any $R \leq R_0/2$, if $u$ is a weak solution of (1.1) in $B_{11R}(\overline{x}) \subseteq \Omega$, the coefficients $a^{ij}_u(x)$ are $(\delta, 8R)$-vanishing in $B_R(\overline{x})$ and

$$
\left| \left\{ x \in B_R(\overline{x}) : \mathcal{M}_{B_{11R}(\overline{x})} \left( |Xu|^2 \right) (x) > N_1^2 \right\} \right| < \varepsilon |B_R(\overline{x})| 
$$

(5.3)

(where $N_1$ is like in Theorem 4.3), then for any positive integer $m$,

$$
\left| \left\{ x \in B_R(\overline{x}) : \mathcal{M}_{B_{11R}(\overline{x})} \left( |Xu|^2 \right) (x) > N_1^{2m} \right\} \right| \leq \sum_{i=1}^{m} \varepsilon_1^i \left| \left\{ x \in B_R(\overline{x}) : \mathcal{M}_{B_{11R}(\overline{x})} \left( |\mathbf{F}|^2 \right) (x) > \delta^2 N_1^{2(m-1)} \right\} \right| 
$$

$$
+ \varepsilon_1^m \left| \left\{ x \in B_R(\overline{x}) : \mathcal{M}_{B_{11R}(\overline{x})} \left( |Xu|^2 \right) (x) > 1 \right\} \right| 
$$

where $\varepsilon_1 = \varepsilon 5^0 / c_d$.

**Proof.** Fix $\varepsilon$, $R_0 > 0$ and pick $\delta = \delta (\varepsilon, R_0, \mu)$ as in Corollary 4.4. We will prove this assertion by induction on $m$. For $m = 1$, we want to apply Lemma 5.3 to

$$
C := \left\{ x \in B_R(\overline{x}) : \mathcal{M}_{B_{11R}(\overline{x})} \left( |Xu|^2 \right) (x) > N_1^2 \right\}, \\
D := \left\{ x \in B_R(\overline{x}) : \mathcal{M}_{B_{11R}(\overline{x})} \left( |\mathbf{F}|^2 \right) (x) > \delta^2 \right\} \cup \left\{ x \in B_R(\overline{x}) : \mathcal{M}_{B_{11R}(\overline{x})} \left( |Xu|^2 \right) (x) > 1 \right\}.
$$

Since $N_1 \geq 1$, $C \subseteq D \subseteq B_R(\overline{x})$. Also, by assumption $|C| < \varepsilon |B_R(\overline{x})|$. Let $x \in B_R(\overline{x})$ such that

$$
|C \cap B_r(x)| \geq \varepsilon |B_r(x)|.
$$

Then by Corollary 4.4

$$
B_r(x) \cap B_R(\overline{x}) \subseteq D;
$$

hence by Lemma 5.3

$$
|C| \leq \frac{\varepsilon 5^0}{c_d} |D|
$$

which is our assertion for $m = 1$.

Now assume that the assertion is valid for some $m$. Let $u$ be a weak solution to (1.1) in $B_{11R}(\overline{x})$ satisfying (5.3). Set $u_1 = u/N_1$ and $\mathbf{F}_1 = \mathbf{F}/N_1$, then $u_1$ is a weak solution of

$$
X_i \left( a^{ij}_u(x) X_j u_1 \right) = X_i \mathbf{F}_1
$$

in $B_{11R}(\overline{x}) \subseteq \Omega$, and satisfies

$$
\left| \left\{ x \in B_R(\overline{x}) : \mathcal{M}_{B_{11R}(\overline{x})} \left( |Xu_1|^2 \right) (x) > N_1^2 \right\} \right| = \left| \left\{ x \in B_R(\overline{x}) : \mathcal{M}_{B_{11R}(\overline{x})} \left( |Xu|^2 \right) (x) > N_1^2 \right\} \right| < \varepsilon |B_R(\overline{x})|.
$$

By the induction assumption on $m$, we have

$$
\left| \left\{ x \in B_R(\overline{x}) : \mathcal{M}_{B_{11R}(\overline{x})} \left( |Xu|^2 \right) (x) > N_1^{2m} \right\} \right| = \left| \left\{ x \in B_R(\overline{x}) : \mathcal{M}_{B_{11R}(\overline{x})} \left( |Xu_1|^2 \right) (x) > N_1^{2m} \right\} \right|
$$

$$
\leq \sum_{i=1}^{m} \varepsilon_1^i \left| \left\{ x \in B_R(\overline{x}) : \mathcal{M}_{B_{11R}(\overline{x})} \left( |\mathbf{F}|^2 \right) (x) > \delta^2 N_1^{2(m-1)} \right\} \right| + \varepsilon_1^m \left| \left\{ x \in B_R(\overline{x}) : \mathcal{M}_{B_{11R}(\overline{x})} \left( |Xu|^2 \right) (x) > 1 \right\} \right| 
$$

$$
= \sum_{i=1}^{m} \varepsilon_1^i \left| \left\{ x \in B_R(\overline{x}) : \mathcal{M}_{B_{11R}(\overline{x})} \left( |\mathbf{F}|^2 \right) (x) > \delta^2 N_1^{2(m-1)} \right\} \right| + \varepsilon_1^m \left| \left\{ x \in B_R(\overline{x}) : \mathcal{M}_{B_{11R}(\overline{x})} \left( |Xu|^2 \right) (x) > N_1^2 \right\} \right|. 
$$

(5.4)

On the other hand, by the assertion valid for $m = 1$,

$$
\left| \left\{ x \in B_R(\overline{x}) : \mathcal{M}_{B_{11R}(\overline{x})} \left( |Xu|^2 \right) (x) > N_1^2 \right\} \right| \leq \varepsilon_1 \left| \left\{ x \in B_R(\overline{x}) : \mathcal{M}_{B_{11R}(\overline{x})} \left( |\mathbf{F}|^2 \right) (x) > \delta^2 \right\} \right|
$$

$$
+ \varepsilon_1 \left| \left\{ x \in B_R(\overline{x}) : \mathcal{M}_{B_{11R}(\overline{x})} \left( |Xu|^2 \right) (x) > 1 \right\} \right|. 
$$

(5.5)
Putting (5.5) into (5.4) we get
\[
\left| \{ x \in B_R(\overline{x}) : M_{\beta_{1+1}}(\|Xu\|^2)(x) > N_1^{2(m+1)} \} \right| \leq \sum_{i=1}^{m} e_i \left| \{ x \in B_R(\overline{x}) : M_{\beta_{1+1}}(\|F\|^2)(x) > \delta^2 N_1^{2(m+1)} \} \right| \\
+ \varepsilon^{m+1} \left| \{ x \in B_R(\overline{x}) : M_{\beta_{1+1}}(\|Xu\|^2)(x) > 1 \} \right| + \varepsilon^{m+1} \left| \{ x \in B_R(\overline{x}) : M_{\beta_{1+1}}(\|F\|^2)(x) > \delta^2 \} \right| \\
= \sum_{i=1}^{m+1} e_i \left| \{ x \in B_R(\overline{x}) : M_{\beta_{1+1}}(\|F\|^2)(x) > \delta^2 N_1^{2(m+1)} \} \right| + \varepsilon^{m+1} \left| \{ x \in B_R(\overline{x}) : M_{\beta_{1+1}}(\|Xu\|^2)(x) > 1 \} \right| 
\]
which is the desired assertion for \( m + 1 \). This completes the proof. \( \square \)

We can finally come to the following.

**Proof of Theorem 2.13.** Fix \( R_0 \), let \( \varepsilon > 0 \) to be chosen later, and pick \( \delta = \delta(\varepsilon, R_0, \mu) \) as in Theorem 5.4. For \( \lambda > 0 \), let \( u_\lambda = \frac{u}{\lambda}, F_\lambda = \frac{F}{\lambda} \). We claim that we can take \( \lambda \) large enough (depending on \( \varepsilon, u \) and \( F \)) so that
\[
\left| \{ x \in B_R(\overline{x}) : M_{\beta_{1+1}}(\|Xu\|^2)(x) > N_1^2 \} \right| < \varepsilon \left| B_R(\overline{x}) \right| 
\]
and
\[
\sum_{k=1}^{\infty} N_1^{kp} \left| \{ x \in B_R(\overline{x}) : M_{\beta_{1+1}}(\|F_\lambda\|^2)(x) > \delta^2 N_1^2 \} \right| < 1. 
\]
Actually, since \( F \in L^p(B_{1+1}(\overline{x})): M_{\beta_{1+1}}(\|F\|^2)(x) \in L^2(B_{1+1}(\overline{x})) \) by Lemma 4.2. Applying Lemma 5.1 with \( f = M_{\beta_{1+1}}(\|F\|^2), \theta = \delta, m = N_1^2, \Omega = B_R(\overline{x}) \) and \( p \) replaced by \( p/2 \), there is a positive constant \( c \) depending only on \( \delta, p, \) and \( N_1 \), such that
\[
\sum_{k=1}^{\infty} N_1^{kp} \left| \{ x \in B_R(\overline{x}) : M_{\beta_{1+1}}(\|F_\lambda\|^2)(x) > \delta^2 N_1^2 \} \right| \leq c \left( M_{\beta_{1+1}}(\|F\|^2) \right)^{p/2}_{L^2(B_{1+1}(\overline{x}))}. 
\]
Also, by Lemma 4.2 we have
\[
\left| \{ x \in B_R(\overline{x}) : M_{\beta_{1+1}}(\|Xu\|^2)(x) > N_1^2 \} \right| = \left| \{ x \in B_R(\overline{x}) : M_{\beta_{1+1}}(\|Xu\|^2)(x) > \lambda^2 N_1^2 \} \right| < \frac{c}{\lambda^2 N_1^2} \|Xu\|^2_{L^2(B_{1+1}(\overline{x}))}. 
\]
Hence we can take
\[
\lambda = c \left( \frac{\|Xu\|^2_{L^2(B_{1+1}(\overline{x})): M_{\beta_{1+1}}(\|Xu\|^2)}}{\varepsilon^{1/2} \left| B_R(\overline{x}) \right|^{1/2}} + \|F\|_{L^p(B_{1+1}(\overline{x}))} \right) 
\]
for some constant \( c \) depending on \( \delta, p, N_1 \); hence \( c = (\varepsilon, R_0, \mu) \), and get (5.6) and (5.7) satisfied. Next, by (5.6) we can apply Theorem 5.4 to \( u_\lambda \) for this large \( \lambda \), writing
\[
\sum_{k=1}^{\infty} N_1^{kp} \left| \{ x \in B_R(\overline{x}) : M_{\beta_{1+1}}(\|Xu\|^2)(x) > N_1^2 \} \right| \\
\leq \sum_{k=1}^{\infty} N_1^{kp} \left( \sum_{i=1}^{k} e_i \left| \{ x \in B_R(\overline{x}) : M_{\beta_{1+1}}(\|F_\lambda\|^2)(x) > \delta^2 N_1^{2(k-1)} \} \right| \\
+ \varepsilon^k \left| \{ x \in B_R(\overline{x}) : M_{\beta_{1+1}}(\|Xu\|^2)(x) > 1 \} \right| \right) \\
= \sum_{i=1}^{\infty} (N_1^{p} e_i) \sum_{k=1}^{\infty} N_1^{p(k-1)} \left| \{ x \in B_R(\overline{x}) : M_{\beta_{1+1}}(\|F_\lambda\|^2)(x) > \delta^2 N_1^{2(k-1)} \} \right| \\
+ \sum_{i=1}^{\infty} (N_1^{p} e_i) \left| \{ x \in B_R(\overline{x}) : M_{\beta_{1+1}}(\|Xu\|^2)(x) > 1 \} \right| \\
\leq \sum_{i=1}^{\infty} (N_1^{p} e_i)^\lambda \left| \{ x \in B_R(\overline{x}) \} \right| < 1 + \left| B_R(\overline{x}) \right|
taking  $\varepsilon$  so that $N^p \varepsilon_1 = 1/2$. We have finally chosen $\varepsilon$ small enough, depending on  $p$  and  $G$ , and a corresponding  $\delta = \delta (\varepsilon, R_0, \mu) = \delta (p, G, R_0, \mu)$. Therefore we can apply Lemma 5.1 to $f = M_{B_1(\mathfrak{g})} \left( | Xu |^2 \right)$  and $m = N^p \varepsilon_1$ getting

$$
\| M_{B_1(\mathfrak{g})} \left( | Xu |^2 \right) \|_{L^{p/2}(\mathfrak{g})} \leq c \left( 1 + R^0 \right)
$$

with $c = c (p, G)$ , which by (5.8) implies

$$
\| M_{B_1(\mathfrak{g})} \left( | Xu |^2 \right) \|_{L^{p/2}(\mathfrak{g})} \leq c \left( \| Xu \|_{L^2(\mathfrak{g})} + \| F \|_{L^p(\mathfrak{g})} \right)
$$

with $c = c (R, R_0, p, G)$ , and recalling that $| f (x) | \leq M_{B_1(\mathfrak{g})} (f) (x)$  for a.e. $x$ , we get

$$
\| Xu \|_{L^p(\mathfrak{g})} \leq c \left( \| Xu \|_{L^2(\mathfrak{g})} + \| F \|_{L^p(\mathfrak{g})} \right)
$$

This completes the proof of Theorem 2.13. □

References


