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Green’s Function and Potential Theory
for the Schrödinger Operator: a Nonprobabilistic Approach (**)
a fundamental estimate for the Green's function of $-A + V$ on $C^{1,1}$ domains. More recently, Cranston-Fabes-Zhao [4], using the results of Caffarelli-Fabes-Mortola-Salsa [2], extended many of these results to the operator $A + V$ on Lipschitz domains, and developed a potential theory for it. On the other hand, Chiarenza-Fabes-Garofalo [3], with a non probabilistic approach, proved for $L$ a Harnack's inequality and the continuity of solutions of $Lu = 0$. Following this line, our aim is to develop, by typical methods of P.D.E., a theory for the operator $L$.

Since, in the usual variational theory, $V$ is assumed in $L^p(\Omega)$ with $p > n/2$, a smaller class than $K(\Omega)$, we shall at first prove—section 1—that Dirichlet's problem for $L$ can still be formulated, and is well posed, if $V$ is, in a proper sense, rather small.

Then—section 2—we shall investigate properties of the Green's function for $L$; particularly, we will show (Th. 2.6) that it is controlled from above and from below by the Green's function for $A$. A «weak theory» for the operator $L$ will be also developed, extending to $L$ some results of [6] about $A$.

In sections 3-4 we shall obtain some potential theoretical results, i.e. properties of solutions of $Lu = 0$. First we shall state—section 3—solvability of Dirichlet's problem for $L$ when the datum is a continuous function defined only on the boundary; from this fact will follow existence of the $L$-harmonic measure. Then we shall obtain an estimate about harmonic measures (Th. 3.8), analogue to the one involving the Green's functions for $L$ and $A$. From this fact a comparison principle for positive solutions of $Lu = 0$, vanishing on a part of the boundary, will follow (section 4). Then we shall state regularity of boundary points for $L$. Finally a «Fatou's theorem» will be obtained, i.e. existence of nontangential boundary limits (a.e. with respect to the $L$-harmonic measure) for positive solutions of $Lu = 0$.

Note that the two facts we have borrowed from [4] (Theorems 1.4 and 4.5) have analytical proofs. So our work relies on no probabilistic argument.

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1. SOME DEFINITIONS AND KNOWN RESULTS.

DIRICHELET'S PROBLEM FOR $L$.

Let $\Omega$ be a bounded Lipschitz domain of $\mathbb{R}^n$ ($n \geq 3$). This means that there exists a pair of positive numbers $r_0$ and $M$ such that for every $z \in \partial \Omega$, local coordinates can be selected so that $B(z, r_0) \cap \partial \Omega$ is the graph of a Lipschitz function $\varphi$ with $|D\varphi| < M$. The constants $r_0$ and $M$ determine what will be called the Lipschitz character of $\Omega$.

The operator $A$ is supposed to have bounded, measurable, real valued coefficients $a_{ij}$. We also suppose $a_{ij} = a_{ji}$ and $A$ uniformly elliptic. So there is a positive constant $\lambda$ such that

$$\lambda^{-1} |\xi|^2 < a_{ij}(x) \xi_i \xi_j < \lambda |\xi|^2$$

for every $\xi \in \mathbb{R}^n$, for a.e. $x \in \Omega$. 

Let us now define the Kato class $K(\Omega)$.

\[
K(\Omega) = \left\{ f \in \mathcal{C}_{\text{loc}}^1(\Omega) : \lim_{r \downarrow 0} \sup_{x \in \mathbb{R}^n} \int_{\Omega \cap B(x,r)} \frac{|f(y)|}{|x-y|^{n-2}} \, dy = 0 \right\}.
\]

If $V \in K(\Omega)$, put

\[
\eta(r) = \sup_{x \in \mathbb{R}^n} \int_{\Omega \cap B(x,r)} \frac{|V(y)|}{|x-y|^{n-2}} \, dy.
\]

Sometimes we shall call $\eta$ «the Kato norm of $V$». Now, all the informations we need about $L$ are contained in $\lambda$ and $\eta$.

Note that if $V \in L^p(\Omega)$ with $p > n/2$, then by Hölder’s inequality, $V \in K(\Omega)$ and $\eta(r) \leq c \cdot \|V\|_p \cdot r^\alpha$ with $c$, $\alpha$ only depending on $n$ and $p$. So our assumptions generalize the case which is studied in standard variational approach.

If $V \in K(\Omega)$, it is easy to prove the following properties:

(i) $V \in \mathcal{C}^1(\Omega)$;

(ii) $\eta(r)$ is finite for every $r$, monotone non decreasing;

(iii) $\|V\|_1 \leq d^{n-2} \cdot \eta(d)$ where $d = \text{diam} \, \Omega$;

(iv) $\sup_{\Omega} \int_{\Omega} \frac{|V(y)|}{|x-y|^{n-2}} \, dy \leq \eta(d)$;

(v) $\eta$ is bounded and definitively constant, $\eta(r) < \eta(2d)$ for every $r$;

(vi) if $f(x) = \int_{\Omega} \frac{|V(y)|}{|x-y|^{n-2}} \, dy$, $f$ is continuous in $\Omega$.

Note that, if $f \in \mathcal{C}^1_{\text{loc}}(\Omega)$ and $\eta(r) < \infty$ for some $r$, then properties (i)-(v) hold, but $f$ must not necessarily belong to $K(\Omega)$. (A counterexample is given in [1]). So the crucial property in defining $K(\Omega)$ is that $\eta(r) \to 0$.

A fundamental result, due to Schechter (See [8], p. 138) is the following:

Theorem 1.1: If $V \in K(\Omega)$, there exists a constant $K = K(n)$ and, for every $\delta > 0$, a constant $c_\delta = c_\delta(\delta, n)$, such that for all $\varphi \in W^{1,2}_0(\Omega)$:

\[
\int_{\Omega} |V| |\varphi|^2 < K \cdot \eta(\delta) \|\varphi\|_{W^{1,2}}^2 + c_\delta \cdot \eta(1) \cdot \|\varphi\|_{L^2}^2.
\]

Remark 1.2: From Theorem 1.1 it follows that, since $\Omega$ is Lipschitz, the bilinear form associated to $L$ is well defined and continuous on $W^{1,2}(\Omega)$, and it is coercive on $W^{1,2}_0(\Omega)$ provided that a condition

\[
(1.1) \quad c_1(n, \lambda) \cdot \eta(2d) < 1
\]

holds, with $d = \text{diam} \, \Omega$. 
Let us recall also two basic estimates regarding the Green’s function $G$ for $A$.

**Theorem 1.3 (see [6]):**

\begin{equation}
G(x, y) \leq \frac{c_2}{|x - y|^{n-2}} \quad \text{for all } x, y \in \Omega, \text{ for some constant } c_2(n, \lambda) .
\end{equation}

**Theorem 1.4 (see [4]):** There exists a constant $c_3$ depending on $\lambda$, $n$ and the Lipschitz character of $\Omega$ such that:

\begin{equation}
\frac{G(x, y)G(y, z)}{G(x, z)} \leq c_3 \left\{ \frac{1}{|x - y|^{n-2} + 1} \right\} \quad \text{for all } x, y, z \in \Omega .
\end{equation}

Now, put $c = \max(c_1, c_2, 2c_3)$ where $c_1, c_2, c_3$ are as in (1.1), (1.2), (1.3). Note that $\epsilon = \epsilon(\lambda, n, r_0, M)$. Put:

\begin{equation}
\delta = \epsilon \cdot \eta(2d) .
\end{equation}

Henceforth we shall suppose the Kato norm of $V$ so small to have:

\begin{equation}
\delta < \frac{1}{2} .
\end{equation}

Then (1.1) holds, and Lax-Milgram’s lemma implies:

**Theorem 1.5:** The problem

\[
\begin{cases}
Lu = T & \text{in } \Omega , \\
\quad u = g & \text{on } \partial \Omega ,
\end{cases}
\]

for $T \in W^{-1,2}$ and $g \in W^{1,2}$ assigned, is well posed. The constants in the continuous dependence estimate depend on $n$, $\lambda$, $\eta$, $r_0$, $M$.

(If $g \equiv 0$, the constant only depends on $n$, $\lambda$, $\eta$.)

Let us state also a maximum principle for $L$. From coerciveness of the bilinear form associated to $L$ it follows:

**Theorem 1.6:** If $u \in W^{1,2}(\Omega)$ is a supersolution for $L$ and $u \geq 0$ on $\partial \Omega$ (in sense $W^{1,2}$) then $u \geq 0$ a.e. in $\Omega$.

2. - GREEN’S FUNCTION FOR $L$

In the following, we shall indicate with $G$ and $G_L$ the Green’s functions for $A$ and $L$, respectively. Existence of $G_L$ will follow from theorem 2.3. First, we state a lemma regarding $G$.  

Lemma 2.1. Let \( f \in L^1(\Omega) \) such that \( \sup_{x, y} |G(x, y)| |f(y)| \, dy < \epsilon < \infty \) for some constant \( \epsilon \). Then there exists a unique \( u \in W^{1,2}_0(\Omega) \) satisfying \( Au = f \) in \( \Omega \), in the sense that

\[
\int_{\Omega} a_{ij}(x) u_{x_i} u_{x_j} \, dx = \int_{\Omega} f \varphi \, dx \quad \text{for all } \varphi \in C_0^\infty(\Omega).
\]

Moreover \( u(x) = \int_{\Omega} G(x, y) f(y) \, dy \).

**Proof:** Put

\[
f_n(x) = \begin{cases} 
  n, & \text{if } f(x) > n, \\
  f(x), & \text{if } |f(x)| < n, \\
  -n, & \text{if } f(x) < -n,
\end{cases}
\]

and let \( u_n \) be the solution of

\[
\begin{cases}
  Au_n = f_n & \text{in } \Omega, \\
  u_n \in W^{1,2}_0(\Omega).
\end{cases}
\]

Then

\[
u_n(x) = \int_{\Omega} G(x, y) f_n(y) \, dy \quad \text{and} \quad ||u_n||_{\infty} < \epsilon.
\]

From the equation \( Au_n = f_n \) it follows:

\[
\int_{\Omega} |Du_n|^2 \, dx < \lambda \int_{\Omega} a_{ij}(x) u_{x_i} u_{x_j} \, dx = \lambda \int_{\Omega} u_n f_n \, dx \leq \lambda ||u_n||_{1} ||f_n||_{1} < \lambda \epsilon ||f||_{1}.
\]

Therefore \( \{u_n\} \) is bounded in \( W^{1,2}_0(\Omega) \) and there exists a subsequence \( u_n \) converging to some \( u \) weakly in \( W^{1,2}_0 \) and strongly in \( L^2 \). So, taking limits in:

\[
\int_{\Omega} a_{ij}(x) u_{x_i} \varphi_{x_j} \, dx = \int_{\Omega} f \varphi
\]

for a fixed \( \varphi \in C_0^\infty(\Omega) \), we have:

\[
\int_{\Omega} a_{ij} u_{x_i} \varphi_{x_j} \, dx = \int f \varphi.
\]

On the other hand, choosing a subsequence \( f_n \rightarrow f \) a.e., since

\[
G(x, \cdot) |f_n| \leq G(x, \cdot) |f| \in L^1,
\]
by Lebesgue's theorem one has:

\[ u_n(x) \to \int_\Omega G(x, y) f(y) \, dy \]

that is

\[ u(x) = \int_\Omega G(x, y) f(y) \, dy \]

Remark 2.2: When \( f = V \in K(\Omega) \), by (1.3) and definition of \( K(\Omega) \) we have:

\[ \int_\Omega G(x, y) |V(y)| \, dy \leq \epsilon \int_\Omega \frac{|V(y)|}{|x-y|^{n-2}} \, dy \leq \epsilon \eta(d) < \delta , \]

and the previous lemma holds. Moreover, by Theorem 1.1, the functional \( \varphi \rightarrow \int_\Omega V \varphi \) is continuous on \( \mathcal{W}^{1,2}_0 \). So we have:

\[ \int_\Omega a_{ij} u_{x_i} u_{x_j} = \int_\Omega V \varphi \quad \text{for all} \quad \varphi \in \mathcal{W}^{1,2}_0 . \]

Next theorem is taken from [3].

Theorem 2.3: If \( u \in \mathcal{W}^{1,2}_0 \) is the solution of \( Lu = f \) with \( f \in L^p(\Omega) \), \( p > n/2 \), then:

\[ \| u \|_{\infty} < \epsilon \| f \|_p \quad \text{with} \quad \epsilon = \frac{c(n, \lambda, p, |\Omega|)}{1 - \delta} . \]

Remark 2.4: As a consequence of Theorem 2.3, there exists the Green's function \( G_L \) for \( L \), i.e. for every \( f \in L^p \) (\( p > n/2 \)), the solution of:

\[ \begin{cases} 
  Lu = f , \\
  u \in \mathcal{W}^{1,2}_0 ,
\end{cases} \]

is given by:

\[ u(x) = \int_\Omega G_L(x, y) f(y) \, dy \]

with:

\[ \sup_x \| G_L(x, \cdot) \|_q < c(n, \lambda, q, |\Omega|, \delta) \quad \text{for all} \quad q < \frac{n}{n-2} . \]

Moreover \( G_L \) is symmetric and, by the maximum principle (Theorem 1.6), nonnegative.
The maximum principle also implies the following:

**Corollary 2.5:** Let \( V_1, V_2 \in K(\Omega) \) (both \( V_i \) satisfying our assumption \( \delta < \frac{1}{2} \)) and let \( G_{L_1}, G_{L_2} \) be the Green's functions for \( A + V_1, A + V_2 \), respectively. If \( V_1 \leq V_2, G_{L_1} \leq G_{L_2} \).

Now we can state our basic result:

**Theorem 2.6 (Comparison between \( G \) and \( G_L \))**:

\[
(1 - 2\delta) \cdot G(x, y) < G_L(x, y) \leq \frac{1}{1 - \delta} \cdot G(x, y)
\]

for a.e. \( x, y \in \Omega \).

**Proof:** Using the representation formula (2.2) one can find the following identity:

\[
G_L(x, y) = G(x, y) - \int_{\Omega} G_L(x, w)V(w)G(w, y)\, dw
\]

for a.e. \( x, y \in \Omega \).

Now, let us consider the space \( B \) defined by:

\[
B = \left\{ f : \Omega \times \Omega \to \mathbb{R}, \text{ } f \text{ measurable such that } \| f \|_B = \sup_{x, y} \left| f(x, y) \right| dy < +\infty \right\}.
\]

\( B \) is a Banach space. If we define the operator \( T \) as:

\[
Tf(x, y) = \int_{\Omega} f(w, y)V(w)G(x, w)\, dw
\]

it is easy to verify that \( T \) is a well defined, linear continuous operator from \( B \) to \( B \), with \( \| T \|_{\mathcal{L}(B)} < \delta \). Let us consider the integral equation:

\[
f + Tf = G
\]

where the unknown function \( f \) is sought in \( B \). Then \((I + T)\) can be inverted by Neumann series:

\[
(I + T)^{-1} = \sum_{0}^{\infty} (-)^n T^n
\]

(where the series converges in \( \mathcal{L}(B) \)). Since, by (2.5), the solution of (2.6) is \( G_L \), (2.7) gives:

\[
G_L = \sum_{0}^{\infty} (-)^n T^n G
\]
(where the series converges in $B$). Using Theorem 1.4 we have:

$$
|TG(x, y)| = \left| \int_B G(w, y) V(w) G(x, w) dw \right| < \\
< G(x, y) \cdot \left\{ \int_B \frac{G(x, w) G(w, y)}{G(x, y)} |V(w)| dw < G(x, y) \cdot \epsilon_3 \cdot \epsilon_3 \cdot 2\eta(d) < \delta \cdot G(x, y) \right\} \cdot \left\{ \int_B \frac{|V(w)|}{|y - w|^{n-2}} dw + \int_B \frac{|V(w)|}{|x - w|^{n-2}} dw \right\} \leq G(x, y) \cdot \epsilon_3 \cdot 2\eta(d) < \delta \cdot G(x, y) .
$$

By iteration:

$$
T^n G(x, y) | < \delta^n \cdot G(x, y).
$$

(2.9)

Since convergence in $B$ implies convergence a.e. of a subsequence, it follows from (2.8) and (2.9) that:

$$
G_L(x, y) = \frac{1}{1 - \delta} \cdot G(x, y) \quad \text{for a.e. } x, y \in \Omega,
$$

and so we have the right hand inequality in (2.4).

On the other hand, again from (2.5) we have:

$$
G_L(x, y) = G(x, y) - \int_B G_L(w, y) V(w) G(x, w) dw = \\
= G(x, y) \cdot \left\{ 1 - \int_B \frac{G_L(w, y) G(x, w)}{G(x, y)} |V(w)| dw \right\} > \\
G(x, y) \cdot \left\{ 1 - \frac{1}{1 - \delta} \int_B \frac{G(w, y) G(x, w)}{G(x, y)} |V(w)| dw \right\} \quad \text{(by (2.10))}
$$

(2.10)

and the proof is complete. //

Let us see some consequences of Theorem 2.6. Combining this fact with results in [6], we have:

**Theorem 2.7:** Let $\Sigma$ be a simply connected, bounded Lipschitz domain which can be mapped smoothly onto a sphere, and let $G_L, g$ be the Green's functions for $L$ and $-\Delta$, respectively, in $\Sigma$. Then, for any compact subset $C$ of $\Sigma$, there exists a constant $k$ only depending on $C, \Sigma$ and $\delta$, such that:

$$
k^{-1} \cdot g(x, y) < G_L(x, y) < k \cdot g(x, y) \quad \text{for a.e. } x, y \in C.
$$
Corollary 2.8:

\[ G_L(x, y) \leq \frac{c(\delta)}{|x - y|^{n-2}} \quad \text{for a.e. } x, y \in \Omega. \]

Remark 2.9: We can obtain from (2.11) an integrability property of \( G_L \) (and \( G \)). Let us recall that \( f \in \mathcal{L}^{p, \infty}(\Omega) \) if, by definition, \( f \in \mathcal{L}^{1}_{\text{loc}}(\Omega) \) and:

\[
\| f \|_{p, \infty} = \sup_{K} \frac{\int_K |f(x)| \, dx}{|K|^{1/p}} < +\infty
\]

where the sup is taken among all compact subsets of \( \Omega \) and \( p^{-1} + q^{-1} = 1 \). It is clear that \( \mathcal{L}^p(\Omega) \) is continuously embedded in \( \mathcal{L}^{p, \infty}(\Omega) \). We already know that \( G_L(x, \cdot) \in \mathcal{L}^q \) for \( q < n/(n-2) \). Since it is known that the function \( f(x) = |x - y|^{2-n} \) belongs to \( \mathcal{L}^{1/(n-2), \infty} \), uniformly in \( x \), by (2.11) the same is true for \( G_L \):

\[
\sup_{x} \| G_L(x, \cdot) \|_{n/(n-2), \infty} < c(\delta).
\]

Remark 2.10: Let \( V = V^+ - V^- \), let \( \eta \) be the Kato norm of \( V^- \), \( \eta \) satisfying our assumptions (1.4)-(1.5), and \( V^+ \in K(\Omega) \). Then the bilinear form associated to \( L \) is still coercive, and there exists the Green’s function \( G_L \). By Corollary 2.5, \( G_L \leq G_{A-V^-} \), while \( G_{A-V^-} \) clearly satisfies Theorem 2.7. So it is still true that

\[
G_L(x, y) \leq \frac{1}{1-\delta} \cdot G(x, y).
\]

Green’s function can be seen also as the solution of \( LG_L(x, \cdot) = \delta_x \) (where \( \delta_x \) is the Dirac mass concentrated in \( x \)) in a weak sense, by developing a «weak theory» for the equation \( Lu = \mu \) (where \( \mu \) is a measure). This can be done, following [6], by transferring to \( L \) some properties which are known to hold for \( A \).

Theorem 2.11: If \( u \) is the solution of

\[
\begin{align*}
L u &= (f_i)_{x_i} \quad \text{in } \Omega, \\
 u &= b \quad \text{on } \partial \Omega,
\end{align*}
\]

with \( f_i \in \mathcal{L}^p(\Omega) \), \( b \in \mathcal{W}^{1,p}(\Omega) \) and \( p > n \), then \( u \) is continuous on \( \bar{\Omega} \).

Theorem 2.12. Under the same assumptions of the previous theorem, if \( b \equiv 0 \) then

\[
\max_{\Omega} |u(x)| \leq c(n, \lambda, p, |\Omega|, \delta) \cdot \| f_i \|_p.
\]
Proof: Theorems 2.11-2.12 hold for \( A \); these results are due to Stampacchia (see [6] for references). Since \( u \) is bounded, \( V u \in K(\Omega) \). So, by Lemma 2.1 and Remark 2.2, \( u \) can be seen as the sum of a function satisfying (2.12) with \( L \) substituted by \( A \), and a function expressed by:

\[
\tilde{u}(x) = -\int_{\Omega} G(x, y) V(y) u(y) \, dy.
\]

Now, \( \tilde{u} \) is continuous, by definition of Kato class and (1.2), so Theorem 2.11 is proved. Moreover:

\[
\|u\|_{\infty} \leq \epsilon \cdot \|f_i\|_p + \delta \cdot \|u\|_{\infty}
\]

and Theorem 2.12 follows. //

Definition 2.13: For a measure \( \mu \) of bounded variation on \( \Omega \), we say that \( u \in \mathcal{L}^1(\Omega) \) is a weak solution of the equation \( Lu = \mu \) vanishing at the boundary \( \partial \Omega \) if it satisfies

\[
\int_{\Omega} u \cdot L \varphi \, dx = \int_{\Omega} \varphi \, d\mu
\]

for every \( \varphi \in W^{1,2}_0(\Omega) \cap C(\overline{\Omega}) \) such that \( L \varphi \in C(\overline{\Omega}) \).

Once one knows the results of Theorems 2.11-2.12, the arguments contained in sections 5-6 of [6] can be repeated. We summarize the main results in the following:

Theorem 2.14: For any measure \( \mu \) of bounded variation, a unique solution \( u \) of \( Lu = \mu \) vanishing at \( \partial \Omega \) exists, and lies in \( W^{1,p'}_0(\Omega) \) for every \( p' < n/(n-1) \); moreover \( u \) satisfies

\[
\|u\|_{W^{1,p'}_0} \leq C(n, \lambda, p', |\Omega|, \delta) \int_{\Omega} |d\mu|
\]

and \( u \) is assigned by the integral (a.e. converging)

\[
u(x) = \int_{\Omega} G_{\lambda}(x, y) \, d\mu(y).
\]

Finally, \( G_{\lambda}(x, \cdot) \) is the weak solution vanishing at \( \partial \Omega \) of \( L u = \delta_x \).

Remark 2.15: If \( u \) is the weak solution of \( Lu = \mu \) \((A u = \mu)\) vanishing at \( \partial \Omega \), then \( V u \in \mathcal{L}^1(\Omega) \) and

\[
\| Vu \|_1 \leq \epsilon \int_{\Omega} |d\mu|
\]
with \( \epsilon = \delta/(1 - \delta) \) \((\epsilon = \delta)\). In fact we have:

\[
\int |Vu| < \int |V(x)| \, dx \int G_L(x, y) |d\mu(y)| = \int |d\mu(y)| \int G_L(x, y) |V(x)| \, dx \leq \frac{\delta}{1 - \delta} \int |d\mu|
\]

where we have used (2.1) and (2.4).

Let us point out also the following regularity properties of \( G_L \):

**Theorem 2.16:**

(i) \( G_L(x, \cdot) \in W^{1, p'}(\Omega) \) for every \( p' < n/(n-1) \);

(ii) \( G_L(x, \cdot) \in W^{1, 2}_{\text{loc}}(\Omega - \{x\}) \) and is a local solution of \( Lu = 0 \) in \( \Omega - \{x\} \);

(iii) \( G_L(x, \cdot) \in C(\Omega - \{x\}) \).

**Proof:** (i) follows from 2.14, while (ii)-(iii) can be stated as in [6], using two results about \( L \) (a «Caccioppoli’s inequality» and the continuity of solutions of \( Lu = 0 \)) contained in [3].


3. - Dirichlet’s problem with continuous boundary data.

**L-harmonic measure**

In order to define the concept of \( L \)-harmonic measure and develop a potential theory for \( L \), we need a sharper version of maximum principle.

**Theorem 3.1:** If \( u \) is a supersolution (subsolution) for \( L \) in \( \Omega \), then, respectively:

a) \( \min_{\Omega} u > \frac{1}{1 - \delta} \cdot \min_{\partial \Omega} u_- \),

b) \( \max_{\Omega} u < \frac{1}{1 - \delta} \cdot \max_{\partial \Omega} u^+ \).

If \( u \) is a solution of \( Lu = 0 \) in \( \Omega \), then

\[
\max_{\Omega} |u| < \frac{1}{1 - \delta} \cdot \max_{\partial \Omega} |u|.
\]

**Proof:** Let \( u > k \) on \( \partial \Omega \) for some \( k < 0 \). Then

\[
L(u - k) = Lu - Vk > -Vk \quad \text{in} \ \Omega,
\]

\( (u - k) > 0 \) \quad \text{on} \ \partial \Omega .
Let \( w \) be the solution of
\[
\begin{align*}
Lw &= -Vk & \text{in } \Omega, \\
w &= 0 & \text{on } \partial\Omega.
\end{align*}
\]
Then \( w(x) = -k \int_{\partial \Omega} G(x, y)V(y) \, dy \) and, by (2.1) and (2.4),
\[
|w(x)| \leq -\frac{k}{1-\delta} \int_{\partial \Omega} G(x, y)|V(y)| \, dy \leq -k \cdot \frac{\delta}{1-\delta}.
\]
Put \( v = (u - k) - w \). By the maximum principle (Th. 1.6), \( v \geq 0 \) in \( \Omega \) and so
\[
(3.3) \quad u(x) \geq k - |w(x)| \geq \frac{k}{1-\delta}.
\]
Now, \( \min_{\partial \Omega} u = \sup_{\partial \Omega} \{ k : u \geq k \text{ on } \partial \Omega \} \). If \( \min_{\partial \Omega} u > 0 \), (3.1.a) holds by Theorem 1.6 while if \( \min_{\partial \Omega} u < 0 \), (3.1.a) follows from (3.3).

Changing \( u \) in \( -u \) it follows (3.1.b), and combining (3.1.a) with (3.1.b) one has (3.2).

\[\text{Remark 3.2: The constant } \delta \text{ in (3.1)-(3.2) actually depends only on } V^- , \text{ so that when } V > 0 \text{ one has:} \]
\[
(3.4) \quad \max_{\bar{\Omega}} |u| \leq \max_{\partial \Omega} |u|.
\]
In the general case one cannot expect (3.4) to be true. To see this, it is sufficient to consider \( \Omega \) the unit ball, \( V \) a small negative constant, \( A = -\Lambda \) : then the solution \( u \) with boundary value 1 is a positive function assuming a strong maximum at the origin.

Now, using Theorem 2.1 and Caccioppoli's inequality of [3], one can repeat an argument used in [6] and give sense to Dirichelet's problem for \( L \) when the datum is a continuous function defined on \( \partial \Omega \). Namely, the following holds:

**Theorem 3.3:** There exists a mapping \( B \) which to any continuous function \( f \) defined on \( \partial \Omega \) associates a local solution of \( Lu = 0 \) (which is continuous by [3]) such that whenever \( f \) is the trace of a \( C^1(\overline{\Omega}) \) function, \( Bf \) coincides with the variational solution of
\[
\begin{align*}
Lu &= 0 & \text{in } \Omega, \\
\ u &= f & \text{on } \partial \Omega.
\end{align*}
\]
Moreover, if \( u = Bf \), one has:
\[
\sup_{K \subseteq \Omega} \text{dist}(K, \partial \Omega) \cdot \| Du \|_{C(K)} + \max_{\partial \Omega} |u| < + \infty;
\]
\[
\max_{\partial \Omega} |u| < \frac{1}{1-\delta} \cdot \max_{\partial \Omega} |f|.
\]

This theorem makes it possible to give the following:

**Definition 3.4:** A point \( y \in \partial \Omega \) is said to be *regular* for \( L \) iff for every \( h \in C(\partial \Omega) \) one has:
\[
\lim_{x \to y \in \partial \Omega} Bb(x) = b(y).
\]

**Definition 3.5:** For a fixed \( x \in \Omega \), let us consider the functional \( f \to Bf(x) \) defined on \( C(\partial \Omega) \). By Riesz’ theorem there exists a positive regular Borel measure \( w_{L}^{x} \) representing it:
\[
Bf(x) = \int_{\partial \Omega} f(y) \, dw_{L}^{x}(y).
\]

We call \( w_{L}^{x} \) the \( L \)-harmonic measure evaluated at \( x \). By the Harnack principle of [3] it follows that, given \( x_{1}, x_{2} \in \Omega \), there exists a constant \( c \) such that:
\[
w_{L}^{x_{1}}(E) < c \cdot w_{L}^{x_{2}}(E) \quad \text{for any Borel set } E \subseteq \partial \Omega.
\]

We shall indicate with \( w_{A}^{x} \) the Borel measure obtained with the same construction for the operator \( A \) (\( A \)-harmonic measure evaluated at \( x \)).

The main result we shall prove in this section is the following: there exist constants \( c_{1}, c_{2} \) such that for any Borel set \( E \) and \( x \in \Omega \):
\[
c_{1} w_{A}^{x}(E) < w_{L}^{x}(E) < c_{2} w_{A}^{x}(E).
\]

From this fact and the results contained in [2] it will follow potential theory for \( L \). To obtain (3.5) two facts are used: the estimate (2.4) involving the Green’s functions for \( A \) and \( L \); the notion of kernel function for \( A \) and some relative results contained in [2].

**Definition 3.6:** Fix \( x \in \Omega \), \( \zeta \in \partial \Omega \). A function \( K_{A}^{x}(w, \zeta) \) defined in \( \Omega \) is called a kernel function at \( \zeta \) for the operator \( A \), normalized at \( x \), if:

(i) \( K_{A}^{x}(\cdot, \zeta) \) is a solution of \( Au = 0 \) in \( \Omega \);

(ii) \( K_{A}^{x}(\cdot, \zeta) \in C(\overline{\Omega} - \{\zeta\}) \) and \( \lim_{w \to z, \rho_{w} \to 0,\ \zeta' \in \partial \Omega} K_{A}^{x}(w, \zeta) = 0 \);

(iii) \( K_{A}^{x}(w, \zeta) > 0 \) for each \( w \in \Omega \) and \( K_{A}^{x}(x, \zeta) = 1 \).
For $x$ and $\xi$ fixed, there exists one and only one kernel function $K_A^x(\cdot, \xi)$ and it is:

$$K_A^x(w, \xi) = \frac{d\nu_{y_A}}{dw_{y_A}}(\xi)$$

(Radon-Nikodym derivative of the $A$-harmonic measures).

It can be proved also that, for any $w \in \Omega$, $\xi \in \partial \Omega$, there exists

$$\lim_{\nu \to \omega \in \partial \Omega} \frac{G(w, \xi)}{G(x, y)} = K_A^x(w, \xi).$$

**Theorem 3.7:** For $x \in \Omega$, $\xi \in \partial \Omega$, there exists

$$F(x, \xi) = \lim_{\nu \to \omega \in \partial \Omega} \frac{G_L(x, y)}{G(x, y)}.$$

Moreover $F$ is continuous on $\Omega \times \partial \Omega$, and:

$$F(x, \xi) = 1 - \int_\partial K_A^x(w, \xi) V(w) G_L(x, w) \, dw.$$

**Proof:** Let us consider (2.5), written as:

$$\frac{G_L(x, y)}{G(x, y)} = 1 - \int_\partial \frac{G(w, \xi)}{G(x, y)} \cdot G_L(x, w) V(w) \, dw.$$

By (3.7), what we have to prove is that in (3.10) the limit can be taken under the integral sign. Put:

$$f_r(y) = \int G_L(x, w) \cdot \frac{G(w, \xi)}{G(x, y)} \cdot V(w) \, dw.$$

By Lebesgue's theorem, Theorems 2.4 and 2.6, one has:

$$\lim_{y \to \omega} f_r(y) = \int K_A^x(w, \xi) G_L(x, w) V(w) \, dw$$

and, by definition of Kato class,

$$\lim_{r \to 0} \lim_{y \to \omega} f_r(y) = \int K_A^x(w, \xi) G_L(x, w) V(w) \, dw \quad \text{uniformly in } \xi.$$
On the other hand, for \( r \to 0 \), \( f_r(y) \) converges uniformly to:

\[
\int_{\partial} G(w, y) \cdot G_L(x, w) V(w) \, dw.
\]

So, exchanging the limits it follows (3.9) and the continuity of \( F \).

**Theorem 3.8 (Comparison between harmonic measures):** For each \( x \in \Omega \), \( \zeta \in \partial \Omega \), one has:

\[
dw^*_L(\zeta) = F(x, \zeta) \, dw^*_{\partial}(\zeta).
\]

Moreover, there exist constants \( c_1, c_2 \) depending on \( \delta \) such that:

\[
c_1 \cdot w^*_{\partial}(E) \leq dw^*_L(E) \leq c_2 \cdot w^*_{\partial}(E)
\]

for every Borel set \( E \), \( x \in \Omega \).

**Proof:** Let \( f \in C(\partial \Omega) \). By (3.9):

\[
\int_{\partial} f(\zeta) F(x, \zeta) \, dw^*_L(\zeta) = \int_{\partial} \int_{\partial} f(\zeta) \, dw^*_L(\zeta) - \int_{\partial} \int_{\partial} K^*_{\partial}(w, \zeta) V(w) G_L(x, w) \, dw = \int_{\partial} f(\zeta) \, dw^*_L(\zeta) - \int_{\partial} G_L(x, w) V(w) \, dw = \int_{\partial} f(\zeta) \, dw^*_L(\zeta).
\]

(by Tonelli and (3.6))

Now, let \( v, u \) be the solutions of

\[
\begin{align*}
Lv &= 0 & & \text{in } \Omega, \\
v &= f & & \text{on } \partial \Omega,
\end{align*}
\]

\[
\begin{align*}
Au &= 0 & & \text{in } \Omega, \\
u &= f & & \text{on } \partial \Omega.
\end{align*}
\]

Then the previous identity becomes:

\[
\int_{\partial} f(\zeta) F(x, \zeta) \, dw^*_L(\zeta) = u(x) - \int_{\partial} G_L(x, w) V(w) u(w) \, dw = v(x) = \int_{\partial} f(\zeta) \, dw^*_L(\zeta).
\]

Since this is true for every \( f \in C(\partial \Omega) \), it follows (3.11). Now, by Theorem 3.7, (2.4) implies:

\[
\frac{1 - 2\delta}{1 - \delta} \leq F(x, \zeta) \leq \frac{1}{1 - \delta}
\]

for all \( x \in \Omega \), \( \zeta \in \partial \Omega \).

So (3.12) follows from (3.11).

4. Potential Theory for \( L \)

**Theorem 4.1 (Boundary Harnack principle):** Let \( \zeta_0 \in \partial \Omega \), \( r > 0 \), \( x_r \in \Omega \) such that \( |x_r - \zeta_0| = r \) and dist \((x_r, \partial \Omega) \simeq r \). If \( v \) is a positive solution of
\( L \nu = 0 \) in \( \Omega \), vanishing continuously on \( \partial \Omega \cap B(\xi_0, 2r) \), then

\[
\sup_{B(\xi_0, r)} v < c \cdot v(\xi_0)
\]

for some constant \( c \) depending on \( n, \lambda, \delta \) and the Lipschitz character of \( \Omega \).

**Proof:** Let \( \Omega' = \Omega \cap B(\xi_0, 2r) \). Then \( v \in C(\Omega') \). Let us call \( w^e_L, w^e_A \) the harmonic measures for \( \Omega' \). Then, by the boundary Harnack principle for \( A \) (see [2]) and (3.12), one has, for \( \nu \in B(\xi_0, r) \):

\[
v(\nu) = \int_{\partial \Omega'} v(\xi) \, dw^e_L(\xi) < c \int_{\partial \Omega'} v(\xi) \, dw^e_A(\xi) < \text{const} \int_{\partial \Omega'} v(\xi) \, dw^e_A(\xi) < \text{const} \int_{\partial \Omega'} v(\xi) \, dw^e_L(\xi) = c \cdot v(\xi_0). \]

In the same way the following can be obtained:

**Theorem 4.2 (Comparison principle):** Let \( u, \nu \) be positive solutions of \( L \nu = 0 \) in \( \Omega \), vanishing continuously on \( \partial \Omega \cap B(\xi, 2r) \). Then

\[
\sup_{B(\xi, r)} \frac{u}{\nu} < c \cdot \frac{u}{\nu} (\xi_0).
\]

**Theorem 4.3 (Comparison between solutions of \( L \) and \( A \)):** Let \( u, \nu \) be positive solutions of \( L \nu = 0, A u = 0 \) in \( \Omega \), vanishing continuously on \( \partial \Omega \cap B(\xi, 2r) \). Then

\[
\sup_{B(\xi, r)} \frac{u}{\nu} < c \cdot \frac{u}{\nu} (\xi_0).
\]

Once one knows these results, one can repeat the arguments contained in section 3 of [2]: the kernel function for the operator \( L \) can be defined, and its existence and uniqueness can be proved. Moreover, from the formula

\[
K^e_L(w, \xi) = \frac{dw^e_L(\xi)}{dw^e_L(w)}
\]

it follows, by (3.11), that

\[
K^e_L(w, \xi) = \frac{F(w, \xi) \, dw^e_A(\xi)}{F(\xi, \xi) \, dw^e_A(\xi)} = \frac{F(w, \xi) \, K^e_A(w, \xi)}{F(\xi, \xi)}.
\]

Hence we have that \( K^e_L(w, \xi) \in C(\partial \Omega) \) and:

\[
\epsilon_1 K^e_L(w, \xi) < K^e_L(w, \xi) < \epsilon_2 K^e_A(w, \xi)
\]

for all \( \nu, w \in \Omega, \xi \in \partial \Omega \), for some constants \( \epsilon_1, \epsilon_2 \) depending on \( \delta \).
Now we are interested in stating regularity of boundary points for $L$. From this fact and the previous theorems, by the same arguments of [2], some results about boundary behavior of nonnegative solutions will follow. First, we want to point out a consequence of comparison theorem.

**Theorem 4.4:** Let $u, v$ be positive solutions of $Lu = 0$ in $\Omega$, vanishing continuously on $\partial \Omega \cap B(z_0, 2r_0)$ (for a fixed $z_0 \in \partial \Omega$, $r_0 > 0$). Then the quotient $u/v$ can be extended as a Holder continuous function on $\overline{\Omega} \cap B(z_0, 2r_0)$. (Note that, by [3], the solutions $u, v$ are in general continuous but not Holder).

**Proof:** For every $r > 0$, put $M_r = \sup_{B_r} u/v, m_r = \inf_{B_r} u/v$. Since $v$ and $M_{2r}, v - u$ are positive solutions in $B_{2r}$, by comparison theorem we have:

$$\sup_{B_r} \left( \frac{M_{2r} - u}{v} \right) < c \inf_{B_r} \left( \frac{M_{2r} - u}{v} \right)$$

that is

(4.1) \[ M_{2r} - m_r < c \cdot (M_{2r} - M_r). \]

Considering now $u - m_{2r}v$, with the same reasoning one obtains

(4.2) \[ M_r - m_{2r} < c \cdot (m_r - m_{2r}) \]

and from (4.1), (4.2), with a standard technique (see [7]) Holder continuity of $u/v$ follows. \(/\)

The following lemma is taken from [4].

**Lemma 4.5:** For any $z \in \overline{\Omega}$, $w \in \Omega$, $w \equiv z$

(4.3) \[ \lim_{z, y \to z \atop z, y \in \Omega} \frac{G(x, w)G(w, y)}{G(x, y)} = 0. \]

Also, if $z, z' \in \partial \Omega$, then

(4.4) \[ \lim_{z \to z} \frac{G(x, w)G(w, y)}{G(x, y)} = K(z, w, z') \]

exists and is a continuous function of $(z, z')$ on $\partial \Omega \times \partial \Omega$.

Let us now consider $F(x, y) = G_L(x, y)/G(x, y)$. We have already proved that $F$ can be extended continuously on $\Omega \times \overline{\Omega}$. We now state the following:

**Lemma 4.6:** $F$ can be extended continuously on $\overline{\Omega} \times \overline{\Omega}$. For $z, z' \in \partial \Omega$, it is

(4.5) \[ F(z, z') = 1 - \int_{\partial \Omega} F(w, z')K(z, w, z')V(w)dw. \]

(Note that, since $K(z, w, z) = 0$, $F(z, z) = 1$.)
Proof: By (3.9):

\[
\lim_{z \to z'} F(x, z) = 1 - \lim_{z \to z'} \int \lim_{w \to z} \frac{G(w, y)}{G(x, y)} \cdot G_L(x, w) V(w) dw.
\]

Note that

\[
\lim_{w \to z} \frac{G(w, y)}{G(x, y)} \cdot G_L(x, w) = \lim_{w \to z} \frac{G_L(x, w)}{G(x, y)} \cdot \frac{G(x, w) G(w, y)}{G(x, y)} = F(w, z') K(z, w, z') \quad \text{by (3.8) and (4.4).}
\]

Also, by (1.3), we have that \( \int F(w, z') K(z, w, z') V(w) dw \) converges. Put:

\[
f_r(x) = \int_{|w-z|>r} K^*_A(w, z) G_L(x, w) V(w) dw.
\]

By Lebesgue's theorem, (1.2), (1.3) we see that

\[
\lim_{z \to z'} f_r(x) = \int_{|w-z|>r} F(w, z') K(z, w, z') V(w) dw
\]

and the right hand side is a continuous function of \( z' \), uniformly converging, when \( r \to 0 \), to

\[
\int_{\partial \Omega} F(w, z') K(z, w, z') V(w) dw.
\]

So this is a continuous function. Furthermore, when \( r \to 0 \):

\[
f_r(x) \to \int_{\partial \Omega} K^*_A(w, z) G_L(x, w) V(w) dw
\]

uniformly in \( x \) (again by (1.2), (1.3) and definition of Kato class). So (4.5) holds.

Theorem 4.7 (Regularity of boundary points): For every \( f \in C(\partial \Omega) \), \( z_0 \in \partial \Omega \), when \( x \to z_0 \) \( (x \in \Omega) \)

\[
\int_{\partial \Omega} f(z) dw^*_L(z) \to f(z_0).
\]

Proof: Let \( \{x_n\} \) be a sequence in \( \Omega \) converging to \( z_0 \), \( g_n(z) = f(z) \cdot F(x_n, z) \). Then \( g_n \in C(\partial \Omega) \) and \( \|g_n\|_\infty < c \cdot \|f\|_\infty \). By (3.11):

\[
\int_{\partial \Omega} f(z) dw^*_L(z) = \int_{\partial \Omega} g_n(z) dw^*_A(z) = \int_{\partial \Omega} \left[ g_n - f(z_0, \cdot) \right](z) dw^*_A(z) + \int_{\partial \Omega} f(z) F(z_0, \cdot) dw^*_A(z).
\]
The first term tends to zero by uniform continuity of \( F(\cdot, \cdot) \) on \( \partial \Omega \times \partial \Omega \), while the second term, by regularity of boundary points for \( A \) (see [6]) converges to \( f(\zeta_0) F(\zeta_0, \zeta_0) = f(\zeta_0) \) (by (4.5)). So we are done. ∎

From the facts we have stated up to this point, the arguments contained in section 4 of [2] can be repeated for the operator \( L \). Let \( \Sigma \) be the unit ball in \( \mathbb{R}^n \), \( K_L(\cdot, \cdot) \) the kernel function for \( L \) in \( \Sigma \) evaluated at the origin. Then the following hold:

**Theorem 4.8**: Let \( u \) be a nonnegative solution of \( Lu = 0 \) in \( \Sigma \). Then there exists a finite Borel measure \( \nu \) on \( \partial \Sigma \) such that:

\[
(4.6) \quad u(x) = \int_{\partial \Sigma} K_L(x, \zeta) \, d\nu(\zeta).
\]

**Theorem 4.9** (Existence of nontangential limits): Let \( u \) be a nonnegative solution of \( Lu = 0 \) in \( \Sigma \). Then almost everywhere on \( \partial \Sigma \) with respect to the \( L \)-harmonic measure \( \nu_L = \nu_L^0 \), the nontangential limit of \( u \) exists.

If \( \nu \) is as in (4.6), let us consider the Lebesgue decomposition of \( \nu \) with respect to \( \nu_L \):

\[
d\nu = d\nu_s + f \, d\nu_L.
\]

Then the limit is given by \( f \). Moreover, if \( f \) is bounded, the following representation holds:

\[
(4.7) \quad u(x) = \int_{\partial \Sigma} K_L(x, \zeta) f(\zeta) \, d\nu_L(\zeta) = \int_{\partial \Sigma} f(\zeta) \, d\nu_L(\zeta).
\]

As in [2], Theorems 4.8-4.9 still hold when \( \Sigma \) is a bounded Lipschitz star-shaped domain.

**REFERENCES**


