The Spectrum of Projective Curves

by

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Introduction

Our general area of interest is the study of algebraic curves in $\mathbb{P}^3$, the projective space over an algebraically closed field $k$.

In the past fifteen years it has become clear that even if one is primarily interested in smooth irreducible curves, it is often useful to consider more general curves: in studying the relation between curves and reflexive sheaves, Hartshorne [21] was led to consider locally Cohen-Macaulay curves which are generically local complete intersection, while liaison theory as developed in [32] makes it necessary to consider locally Cohen-Macaulay curves, because minimal curves in a given bilaision class need not be reduced, irreducible or generically local complete intersection. We therefore define a curve to be a projective scheme of dimension one which is locally Cohen-Macaulay and equidimensional.

The most important invariants of a curve $C \subseteq \mathbb{P}^3$ are its degree $d$ and its arithmetic genus $g$. The problem of finding the possible pairs $(d, g)$ which occur as the degree and genus of some smooth irreducible curve in $\mathbb{P}^3$ has been solved by Gruson and Peskine [16]; the corresponding problem for locally Cohen-Macaulay equidimensional curves in $\mathbb{P}^3$ turned out to be much simpler and was solved in [19]: there exists a curve $C$ with degree $d$ and arithmetic genus $g$ if and only if $g = 1/2(d - 1)(d - 2)$ or $g \leq 1/2(d - 2)(d - 3)$ (and in the former case $C$ must be a plane curve).

However, the Hilbert scheme parametrizing curves of degree $d$ and genus $g$ is in general very hard to study: it may fail to be reduced and it may have several irreducible components of different dimensions. It is therefore necessary to consider invariants more refined than the degree and the arithmetic genus. In [32], the whole cohomology of the ideal sheaf $\mathcal{I}_C$ of $C$ in $\mathbb{P}^3$ is considered. The sequence of integers

$$h^0(\mathbb{P}^3, \mathcal{I}_C(n)) = \dim H^0(\mathbb{P}^3, \mathcal{I}_C(n))$$
is known as the *postulation* of $C$, the graded module

$$M_C = \bigoplus_{n \in \mathbb{Z}} H^1(\mathbb{P}^3, \mathcal{J}_C(n))$$

is called the *Rao module* of $C$, while the sequence

$$h^1(C, \mathcal{O}_C(n)) = h^2(\mathbb{P}^3, \mathcal{J}_C(n))$$

is called the *speciality* of $C$. Note that the speciality depends only on the curve $C$ and the very ample sheaf $\mathcal{O}(1)$, and not on the particular embedding of $C$ in $\mathbb{P}^3$. Furthermore, by Riemann-Roch to know the speciality of $C$ is equivalent to know the Hilbert function of the graded ring

$$A_C = \bigoplus_{n \in \mathbb{Z}} H^0(C, \mathcal{O}_C(n)).$$

The study of the speciality of $C$ and of the ring $A_C$ is the object of our dissertation. The nice fact about $A_C$ is that it is always a Cohen-Macaulay ring, therefore in some respects easier to investigate than the homogeneous coordinate ring of $C$. The ring structure of $A_C$ depends only on the very ample sheaf $\mathcal{O}_C(1)$, but we are also interested in the extrinsic geometry of the curve in $\mathbb{P}^3$ (for example, in the postulation of the curve), which is reflected by the structure of $A_C$ as a module over the coordinate ring $R = k[x, y, z, w]$ of $\mathbb{P}^3$.

The Hilbert function of $A_C$ is determined by its second difference function $h_C(n)$, which is a finite sequence of nonnegative integers as the following argument shows: if $L$ is a line in $\mathbb{P}^3$ not meeting $C$ and $S_L$ is the polynomial ring generated by the linear forms vanishing on $L$, then $A_C$, being Cohen-Macaulay of dimension two over $R$, is a finitely generated graded free module over $S_L$, and $h_C(n)$ is the number of elements of degree $n$ in a free basis of $A_C$ over $S_L$.

We call the set of integers with multiplicities

$$\text{sp}_C = \{ n^{h_C(n)} \}$$

the *spectrum* of $C$, where the exponent denotes the multiplicity with which $n$ occurs in the set. According to the terminology of [47], the sequence $h_C(n)$ is the $h$-vector of the graded ring $A_C$.

More generally and with no extra effort, we consider those equidimensional locally Cohen-Macaulay subschemes $X \subseteq \mathbb{P}^N$ of *positive* dimension for which the ring

$$A_X = \bigoplus_{n \in \mathbb{Z}} H^0(X, \mathcal{O}_X(n))$$

2
is Cohen-Macaulay. We call such subschemes quasi arithmetically Cohen-Macaulay (for short, quasi ACM). For example, any equidimensional locally Cohen-Macaulay curve in $\mathbb{P}^N$ is quasi ACM. We also say that a zero dimensional subscheme $X \subseteq \mathbb{P}^N$ is quasi ACM, as the homogeneous coordinate ring $R_X$ of a zero dimensional $X$ is always Cohen-Macaulay, and we define $A_X$ to be $R_X$ in this case. With this convention an ACM subscheme is quasi ACM, and a quasi ACM subscheme is locally Cohen-Macaulay and equidimensional.

If $X$ is quasi ACM of dimension $t$, the Hilbert function of $A_X$ is described by its $t+1$ difference function $h_X(n)$, which, as in the case of curves, turns out to be a finite sequence of nonnegative integers; we define the spectrum of $X$ to be the set of integers with multiplicities $\{n^{h_X(n)}\}$.

A Theorem of Macaulay ([8], Theorem 4.2.10, and Theorem 1.6.4 below), makes it possible to describe the set of integers with multiplicities which occur as spectra of arithmetically Cohen-Macaulay subschemes of $\mathbb{P}^N$, but for general quasi ACM subschemes we have little knowledge of what conditions the spectrum must satisfy. Our main theorem in this direction has its origin in corresponding results from the theory of the spectrum for torsion free sheaves on $\mathbb{P}^3$, which has been developed by several authors [5, 21, 22, 42, 43, 46]; in particular, the hard part of the theorem is the analogue of [22], Proposition 5.1, and [42], Theorem 4.1 (which implies most of our statement when $X$ is a curve in $\mathbb{P}^3$). To state our theorem, we need to introduce the Rao module and the index of speciality of a quasi ACM subscheme $X$. The Rao module $M_X$ of $X$ is defined as $A_X/R_X$ where $R_X$ is the homogeneous coordinate ring of $X$. If $X$ has positive dimension, this gives the usual definition:

$$M_X = \bigoplus_{n \in \mathbb{Z}} H^1(\mathbb{P}^N, J_X(n)),$$

while $M_X = 0$ for a zero dimensional $X$. We denote by $\mu(n)$ the number of minimal generators in degree $n$ of the Rao module over the coordinate ring $R$ of $\mathbb{P}^N$:

$$\mu_X(n) = \dim(M_X \otimes_R R/m_R)_n.$$

For a quasi ACM subscheme $X \subseteq \mathbb{P}^N$ of dimension $t$, we define the index of speciality $e(X)$ of $X$ to be the largest integer $n$ such that the graded canonical module $\Omega_X = \Ext^{N-t}_R(A_X, R(-N-1))$ is nonzero in degree $-n$. When $t \geq 1$, this agrees with the usual definition that $e(X)$ be the largest integer $n$ such that $H^0(X, \omega_X(-n))$ is nonzero, where $\omega_X$ denotes the Grothendieck dualizing sheaf of $X$.

**Theorem 0.1 (cf. Theorem 1.7.1)** Let $X \subseteq \mathbb{P}^N$ be a quasi ACM subscheme of dimension $t$ and index of speciality $e = e(X)$. Let $\{n^{h_X(n)}\}$ be
the spectrum of $X$. Then $e + t + 1$ is greater or equal than zero and it is the largest integer occurring in the spectrum of $X$, and the following properties hold:

1. $h_X(n) \geq 1 + \mu_X(n)$ for $0 \leq n \leq e(X) + t + 1$.

2. Suppose there is an integer $l$ satisfying $1 \leq l \leq e + t + 1$ such that $h_X(l) = 1 + \mu_X(l)$. Then
   
   $h_X(n) = 1$ and $\mu_X(n) = 0$ for $l \leq n \leq e + t + 1$.

Furthermore, if $l \leq e + t$, there is a linear subspace $M \subseteq \mathbb{P}^N$ of dimension $t + 1$ such that $X$ contains a hypersurface in $M$ of degree $e + t + 2$.

3. $h_X(n) = \mu_X(n) = 0$ for $n \geq e + t + 2$.

Property 1 in the Theorem implies that the spectrum is connected in positive degrees: more precisely, the nonnegative integers appearing in the spectrum at least once are $0, 1, \ldots, e + t + 1$. Property 2 says in particular that, if $1 \leq n \leq e + t$, then $h_X(n) \geq 2$ unless $X$ has a component contained in a linear space of dimension $t + 1$.

As an application of Theorem 0.1, we obtain a new proof of a result of V. Beorchia \cite{6} bounding the arithmetic genus of a curve $C \subseteq \mathbb{P}^3$ of degree $d$ not contained in a surface of degree $s - 1$. Beorchia’s original proof works only in characteristic zero, while our approach is characteristic free.

**Theorem 0.2 (\cite{6}, Main Theorem, and Theorem 1.8.3 below)** Let $C$ be a curve in $\mathbb{P}^3$ of degree $d$ and genus $g$. Assume that $C$ is not contained in any surface of degree $s - 1$. Then $d \geq s$ and

$$g \leq P(d, s) = \begin{cases} (s - 1)d + 1 - \left(\frac{s + 2}{3}\right) & \text{if } s \leq d \leq 2s \\ \frac{1}{2}(d - s)(d - s - 1) - \left(\frac{s - 1}{3}\right) & \text{if } d \geq 2s + 1 \end{cases}$$

Beorchia \cite{7} has shown that these bounds are sharp for $s \leq 4$, but we don’t know if this is the case for $s \geq 5$.

In chapter 2 we prove a much stronger version of Theorem 0.1 for curves in $\mathbb{P}^3$. If $L$ is a line in $\mathbb{P}^3$, we let

$$\mu_{C,L}(n) = \dim_k(M_C \otimes_{S_L} k)_n$$

denote the number of generators in degree $n$ of the Rao module $M_C$ over the polynomial ring $S_L \cong k[x, y]$ generated by the equations of $L$. Since $S_L$ is a subring of $R$, we have $\mu_{C,L}(n) \geq \mu_C(n)$. Our main result in chapter 2 is the following theorem:
**Theorem 0.3 (cf. Theorem 2.8.2)** Assume the ground field $k$ has characteristic zero. Let $C$ be a curve in $\mathbb{P}^3$ and let $L$ be a general line. Fix an integer $s \geq 1$. If no sub-curve $D$ of $C$ with $e(D) = e(C)$ is contained in a surface of degree $s - 1$, then

$$h_C(n) \geq s + \mu_{C,L}(n) \quad \text{for } s - 1 \leq n \leq e(C) + 3 - s.$$ 

The proof of this theorem uses the theory of (bi)liaison for curves, the Grauert-Mülich Theorem (hence the characteristic zero hypothesis), and the theory of the spectrum for a certain class of torsion free sheaves on $\mathbb{P}^3$.

In [42] a theory of the spectrum is developed for torsion free sheaves on $\mathbb{P}^3$. The spectrum of the ideal sheaf of a curve $C$ as defined in [42] is (up to normalization) the same as the spectrum of $C$ as we have defined it above. However, very little can be said about the spectrum of general torsion free sheaves. We identify a class of torsion free sheaves, which we call curvy sheaves, whose spectrum behaves very much like the spectrum of a curve. A curvy sheaf of rank one is isomorphic to the ideal sheaf of a curve; and in characteristic zero a normalized semistable reflexive sheaf of rank two is a curvy sheaf. Our motivation for studying curvy sheaves is that this is the class of sheaves for which we can prove the following theorem:

**Theorem 0.4 (cf. Theorem 2.7.3)** Let $F$ be a curvy sheaf with spectrum $\{nh_F(n)\}$, and let $L$ be a line on which the splitting type of $F$ is generic. Then

$$h_F(n) \geq \mu_{F,L}(n)$$

where

$$\mu_{F,L}(n) = \dim_k(H^1_*(\mathbb{P}^3, F) \otimes_{S_L} k)_n.$$ 

This theorem is apparently new even for stable vector bundles of rank two, and was suggested by the analogous statement for curves, which is a trivial consequence of the surjection $A_C \to M_C$. For curvy sheaves, following [5] we construct a free graded $S_L$-module $A_{F,L}$, and then we show that it maps surjectively onto $M_F = H^1_*(\mathbb{P}^3, F)$.

An interesting project would be to find an analogue of Theorem 0.3 for sheaves; Theorem 0.4 may be thought of as the case $s = 0$.

As a corollary of the Theorem 0.3, we obtain a version of the Speciality Theorem of Halphen-Gruson-Peskine [15] for Cohen-Macaulay curves, together with a new proof of their result for integral curves:

**Theorem 0.5 (cf. Theorem 2.8.5)** Assume the ground field has characteristic zero. Let $C$ be a curve in $\mathbb{P}^3$ with index of speciality $e$. Assume that no sub-curve $D$ of $C$ with $e(D) = e$ lies on a surface of degree $t - 1$. Then for all integers $m$ satisfying $m \leq \min\{t, (e + 4)/2\}$ we have:
1. \( \deg C \geq m(e + 4 - m) \) with equality holding if and only if \( C \) is a deformation with constant cohomology of a complete intersection of two surfaces of degrees \( m \) and \( e + 4 - m \).

2. \( s(C) \leq \deg C - m(e + 4 - m) + m \), where \( s(C) \) is the smallest degree of a surface containing \( C \).

Furthermore, if \( C \) is integral, then \( s(C) \leq \max\{0, \deg C - n(e + 4 - n)\} + n \) for all integers \( n \).

By taking \( t = m = 1 \), we see that \( s(C) \leq \deg C - e(C) - 2 \) for all Cohen-Macaulay curves (this we can actually prove in any characteristic); curves \( C \) for which \( s(C) \) is actually equal to \( \deg C - e(C) - 2 \) are interesting because they are the only ones which may achieve the maximum genus \( P(d, s) \) prescribed by Beorchia's Theorem. The bound \( s(C) \leq \deg C - e(C) - 2 \) is sharp as for all positive integers \( s \) and \( d \) with \( s \leq d \) there are curves \( C \) satisfying \( s(C) = s \), \( \deg(C) = d \) and \( s = d - e(C) - 2 \). In characteristic zero, Theorem 0.5 makes it possible to describe almost all such curves: if \( C \) is a curve of degree \( d \) satisfying \( s(C) = d - e(C) - 2 \) and \( e(C) \geq 1 \), then \( C \) contains a plane curve of degree \( d - s(C) + 1 \) and a curve \( Y \) with \( s(Y) = \deg Y = s - 1 \); furthermore, the support of \( Y \) is either a line, or the disjoint union of two lines.

In chapter 3 we restrict our attention to the spectrum of an integral quasi ACM subscheme. By Bertini’s Theorem, it is enough to study the spectrum of an integral curve. The problem of determining exactly which sets of integers occur as spectra of integral curves in a particular class has been solved for arithmetically Cohen Macaulay curves in \( \mathbb{P}^3 \) by Gruson and Peskine [15]: they were actually able to determine the possible spectra of integral ACM curves and show that they even occur as spectra of smooth irreducible ACM curves (in characteristic zero; Nollet [39] has shown their result holds in characteristic \( p \) as well). We now translate the result of Gruson and Peskine into our terminology. Given an integer \( s \geq 1 \), an \( s \)-sequence is a sequence of integers \( h(n) \) satisfying the following properties:

\[
\begin{align*}
h(n) &= n + 1 & \text{if } 0 \leq n \leq s - 1 \\
&\geq h(n + 1) & \text{if } s - 1 \leq n \\
&= 0 & \text{if } n \gg 0 \text{ or if } n < 0.
\end{align*}
\]

Given an \( s \)-sequence \( h(n) \), we define \( t \geq s \) to be the least integer \( n \geq s \) such that \( h(t) < s = h(s - 1) \); we say following [31] that \( h(n) \) is of decreasing type if it decreases strictly until it reaches zero for \( n \geq t \). For example, \( \{1, 2, 3, 3, 3, 1\} \) is a 3-sequence of decreasing type with \( t = 5 \), while \( \{1, 2, 3, 2, 2\} \) is a 3-sequence not of decreasing type, with \( t = 3 \).
Theorem 0.6 ([15, 31, 32, 39] cf. Theorem 3.1.3) Given a sequence of non-negative integers \( h(n) \), there is an arithmetically Cohen Macaulay curve \( C \) in \( \mathbb{P}^3 \) such that
\[
h_C(n) = h(n) \quad \text{for all } n \geq 0
\]
if and only if \( h(n) \) is an \( s \)-sequence (with \( s = s(C) \)).

If \( C \) is an integral ACM curve in \( \mathbb{P}^3 \), then \( \{h_C(n)\} \) is of decreasing type; conversely, if \( h(n) \) is an \( s \)-sequence of decreasing type, there is a smooth irreducible ACM curve \( C \) in \( \mathbb{P}^3 \) with \( h_C(n) = h(n) \).

Still, very little is known about the spectrum of a general integral curve (even for ACM curves in higher dimensional projective spaces). Using results of Nollet [39] one can describe the possible spectra of integral curves in several biliaison classes of curves in \( \mathbb{P}^3 \) (for example, the class of a double line of arbitrary arithmetic genus); this generalizes the result of Gruson and Peskine, since a curve in \( \mathbb{P}^3 \) is ACM if and only if it is in the biliaison class of a line.

Stanley has studied the \( h \)-vector of a certain type of graded Cohen-Macaulay domains in [48]. Theorem 3.2.2 below is a version of Theorem 2.1 in [48]; we show that it gives a condition on the spectrum of a semistable reflexive sheaf of rank two on \( \mathbb{P}^3 \) which has not been noticed before (cf. Corollary 3.2.9). [48] also contains a stronger result for integral ACM subschemes, whose proof is based Bertini’s Theorem and on the Uniform Position Theorem of Castelnuovo-Harris. The Uniform Position Theorem is satisfied by all integral curves in characteristic zero, and by most smooth curves in characteristic \( p \) (see [45]); in [13] a similar theorem is proven, weaker, but valid without restrictions in characteristic \( p \). We translate Stanley’s result into our language; and we show that his proof works as well for an integral quasi ACM subscheme:

Theorem 0.7 (Stanley, cf. Theorem 3.3.5) Assume the ground field \( k \) has characteristic zero. Let \( X \) be an integral quasi ACM subscheme of dimension \( t \geq 1 \) with index of speciality \( e \). For all integers \( m, n \) satisfying \( m \geq 0, n \geq 1 \) and \( m + n < e + t + 1 \) we have
\[
\sum_{k=1}^{n} h_X(k) \leq \sum_{k=1}^{n} h_X(m + k).
\]

Stanley’s Theorem gives necessary conditions the spectrum of an integral curve \( C \) must satisfy, and we can show these conditions imply Clifford’s Theorem for the sheaves \( \mathcal{O}_C(l) \) and hence the inequality \((\deg C) e(C) \leq 2g(C) - 2\). On the other hand, these conditions are far from being sufficient.
to guarantee the existence of an integral curve with the given spectrum, and they do not imply Theorem 0.6 for integral ACM curves in $\mathbb{P}^3$. One ingredient missing in Stanley’s Theorem is an understanding of the restrictions the spectrum imposes on the postulation. Consider for example the set of integers with multiplicities

$$sp = \{0, 1^2, 2^2, 3^3, 4\}$$

which satisfies Stanley’s conditions, and suppose it is the spectrum of a curve $C$. Then $h^0\mathcal{O}_C(1) = 4$ and $h^0\mathcal{O}_C(2) = 9$, so that $C$ is a curve contained in some quadric surface in $\mathbb{P}^3$. On the other hand, we can compute the spectra of all integral curves contained in a quadric surface in $\mathbb{P}^3$, and we can check that $sp$ is not among them. Thus there is no integral curve $C$ with spectrum $sp$.

It is therefore interesting to investigate the relation between the spectrum and the postulation of an integral curve $C$ in $\mathbb{P}^3$. It is worth noticing that even the classical problem of finding the maximum genus for a smooth curve of degree $d$ which is not contained on a surface of degree $s - 1$ in $\mathbb{P}^3$ has not been completely solved yet; of course, the degree and genus of a curve $C$ can be computed out of the spectrum, while the invariant $s(C)$ is determined by the postulation, and may vary among curves with a fixed spectrum. Thus a good understanding of the relation between the spectrum and the postulation should bring progress toward a solution of the maximum genus problem; indeed, the strongest results on the maximum genus problem in the difficult range $B$ are in [25], where the relation between the spectrum and the postulation of a rank two reflexive sheaf is studied.

In Section 3.4 we generalize an argument of Hartshorne and Sols [30] to obtain some new conditions the spectrum and the postulation of an integral curve in $\mathbb{P}^3$ must satisfy. These conditions for example imply that a canonical curve of degree 10 in $\mathbb{P}^3$ lies on a quartic surface, as was first shown by Gruson and Peskine [15] (cf. Proposition 3.4.9 below).

In the second half of Chapter 3, we list all possible spectra of integral curves of degree at most 11 lying in some $\mathbb{P}^N$, and we show that they actually occur as spectra of smooth irreducible curves in $\mathbb{P}^3$. On the other hand, we also show that there is a set of integers with multiplicities which is the spectrum of an integral curve of degree 14 in $\mathbb{P}^3$, but which is not the spectrum of any smooth curve.

We consider this work a first step toward a systematic study of the Hilbert schemes of smooth curves of small degrees in $\mathbb{P}^3$: such a study is missing in the modern literature, except for sporadic values of degree and genus. We have available a list of smooth curves of degree $\leq 20$ compiled by Max Noether [38], who shared with G. Halphen the Steiner prize of the
Academy of Berlin in 1882 (!) for the best treatise on the classification of space curves; but Noether’s work has never been checked for completeness and accuracy.

For almost every spectrum $sp$ in our list, we compute the largest value $s_{\text{max}}$ of the invariant $s(C)$ as $C$ varies among integral curves in $\mathbb{P}^3$ having the given spectrum $sp$, and we give an example of a smooth curve $C$ satisfying $sp_C = sp$ and $s(C) = s_{\text{max}}$. Our interest in $s_{\text{max}}$ stems from the importance of the relation between spectrum and postulation which we discussed above.
Chapter 1

The Spectrum of a quasi ACM Subscheme of $\mathbb{P}^N$

1.1 Introduction

The material in this chapter originated from the following remark: if $C$ is a curve (that is, a one dimensional projective scheme with no associated points of dimension zero) in the projective space $\mathbb{P}^N$, the ring

$$A_C = \bigoplus_{n \in \mathbb{Z}} \text{H}^0(C, O_C(n))$$

is Cohen-Macaulay. If $L$ is a codimension two linear subspace which does not meet $C$, projecting from $L$ we obtain a morphism $C \to \mathbb{P}^1$ which corresponds to a finite and flat map of rings $S \to A_C$, where $S \cong k[x, y]$ is the homogeneous coordinate ring of $\mathbb{P}^1$. Therefore $A_C$ is a free finitely generated $S$-algebra and the Hilbert function of $A_C$ is determined by the set of degrees of the elements in a free basis of $A_C$ over $S$; this finite set we call the spectrum of $C$, as it does not depend on $S$. The properties of the spectrum are at the center of our investigation. We don’t need to restrict our attention to curves: we can consider those subschemes $X$ of $\mathbb{P}^N$ which are equidimensional of positive dimension, locally Cohen-Macaulay and for which the ring

$$A_X = \bigoplus_{n \in \mathbb{Z}} \text{H}^0(X, O_X(n))$$

is a Cohen-Macaulay. We call such subschemes quasi arithmetically Cohen-Macaulay, while following the usual terminology we say that $Y \subseteq \mathbb{P}^N$ is
arithmetic Cohen-Macaulay if the homogeneous coordinate ring $R_Y$ of $Y$ is Cohen-Macaulay. If $X$ has dimension zero, then $X$ is automatically ACM, so we say that $X$ is quasi ACM and we set $A_X = R_X$. With this convention, we have:

$$\text{ACM} \Rightarrow \text{quasi ACM} \Rightarrow \text{equidimensional and locally CM}.$$  

We introduce the spectrum for quasi ACM subschemes. That the Hilbert function of certain graded Cohen-Macaulay algebras over a field $k$ may be described by a finite sequence of integers is of course a well known fact: see for example [47] and Chapter 4 in [8]. However, most of the literature deals only with homogeneous $k$-algebras, that is, homogeneous quotients of some polynomial ring. A classical theorem of Macaulay (theorem 4.2.10 in [8]) describes all possible Hilbert functions of homogeneous $k$-algebras, and as a corollary one obtains a characterization of all possible spectra for ACM subschemes of $\mathbb{P}^N$ (proposition 4.3.1 in [8] and corollary 1.6.5 below). For general quasi ACM subschemes, the situation appears to be considerably more complicated. However, we are able to establish some basic properties of the spectrum of an arbitrary ACM subscheme in Theorem 1.7.1. This theorem is modeled after the corresponding result for the spectrum of torsion free sheaves in $\mathbb{P}^3$: the theory of the spectrum for sheaves on $\mathbb{P}^3$ has been developed in [5, 21, 22, 42, 43, 46], and our theorem for curves in $\mathbb{P}^3$ follows (probably) from the results of [42], but even for curves in $\mathbb{P}^N$ it seems new. An interesting project would be to develop a theory of the spectrum from which both the results on quasi ACM subschemes of $\mathbb{P}^N$ and those on torsion free sheaves on $\mathbb{P}^3$ may follow.

As an application of our results on the spectrum, in the last section of this chapter we give a characteristic free proof of a theorem due to V. Beorchia which gives a bound $P(d, t)$ for the arithmetic genus of a curve of degree $d$ in $\mathbb{P}^3$ which is not contained on a surface of degree $t - 1$. Beorchia’s proof in [6] is valid only in characteristic zero.

1.2 Quasi ACM Subschemes of $\mathbb{P}^N$

In this section we introduce the class of quasi arithmetically Cohen-Macaulay subschemes of $\mathbb{P}^N$, for which we will later define the spectrum. The corresponding algebraic notion is that of a Cohen-Macaulay algebra $A$ over the polynomial ring $R = k[x_0, \ldots, x_N]$ with the property that $A$ modulo the image of $R$ has finite length.

We will use the following notation: we will write $\mathbb{P}^N$ for the $N$-dimensional projective space over an algebraically closed field $k$; $R$ will denote the homogeneous coordinate ring of $\mathbb{P}^N$ so that $R \cong k[x_0, \ldots, x_N]$. Given a linear...
subspace \( L \subseteq \mathbb{P}^N \) of codimension \( t + 1 \), we define \( S_L \) to be the polynomial subring of \( R \) generated by the linear forms vanishing on \( L \). We have \( S_L \cong k[y_0, \ldots, y_t] \). If \( \mathcal{F} \) is a quasi-coherent sheaf on a closed subscheme \( X \subseteq \mathbb{P}^N \), we define

\[
H^i(X, \mathcal{F}) = \bigoplus_{n \in \mathbb{Z}} H^i(X, \mathcal{F}(n))
\]

and

\[
h^i(X, \mathcal{F}) = \dim_k H^i(X, \mathcal{F}).
\]

Given a closed subscheme \( X \subseteq \mathbb{P}^N \), we will be looking at the ring \( H^0(\mathbb{P}^N, \mathcal{O}_X) \). This ring is finitely generated over \( R \) if and only if \( H^0(\mathbb{P}^N, \mathcal{O}_X(n)) = 0 \) for \( n \ll 0 \), and this is the case if \( X \) has no associated points of dimension zero:

**Lemma 1.2.1** Let \( X \) be a closed subscheme of \( \mathbb{P}^N \). If \( X \) has no associated point of dimension zero, then

\[
H^0(\mathbb{P}^N, \mathcal{O}_X(n)) = 0 \quad \text{for} \quad n \ll 0.
\]

**Proof** Assume no associated point of \( X \) has dimension zero, and let

\[
r = \dim H^0(\mathbb{P}^N, \mathcal{O}_X)
\]

We claim that \( H^0(\mathbb{P}^N, \mathcal{O}_X(n)) = 0 \) for \( n \leq -r \). If \( \xi \) is an associated point of \( X \), by assumption \( \dim \xi \geq 1 \), hence the set of hyperplanes containing \( \xi \) has codimension at least two in \( (\mathbb{P}^N)^\vee \), the projective space dual to \( \mathbb{P}^N \). Hence for a general line \( D \) in \( (\mathbb{P}^N)^\vee \), every hyperplane in \( D \) contains no associated point of \( X \). Thus, if \( V \) is a general two dimensional space of linear forms in \( R \), for every \( x \in V \) the multiplication map:

\[
x : \mathcal{O}_X(-1) \to \mathcal{O}_X
\]

is injective. Now let \( E \subseteq \mathbb{P}^N \) be the line defined by \( V \). Every homogeneous polynomial in \( S_E = Sym(V) \) is a product of linear forms in \( V \). Hence for every \( \alpha \in H^0(\mathbb{P}^N, \mathcal{O}_X(n)) \) the map \( S_E \to A_X \) which sends one to \( \alpha \) is injective. Now assume by way of contradiction that there is a nonzero element \( \alpha \in H^0(\mathbb{P}^N, \mathcal{O}_X(n)) \) for some \( n \leq -r \). Then

\[
\dim H^0(\mathbb{P}^N, \mathcal{O}_X) \geq \dim(S_E)_{-n} = 1 - n \geq 1 + r
\]

which contradicts the definition of \( r \). Hence \( H^0(\mathbb{P}^N, \mathcal{O}_X(n)) = 0 \) for \( n \leq -r \) and we are done.
We follow the terminology of [8] for the commutative algebra we use. If \((A, m)\) is a Noetherian local ring and \(M \neq 0\) is a finitely generated \(A\)-module, we say that \(M\) is Cohen-Macaulay over \(A\) if \(\text{depth}_m M = \dim M\), where \(\dim M\) is the combinatorial dimension of the ring \(A/\text{ann}(M)\), that is, the affine dimension of the support of \(M\). If furthermore \(\dim M = \dim A\), we say that \(M\) is a maximal Cohen-Macaulay module. In general, if \(A\) is an arbitrary Noetherian ring and \(M\) is a finitely generated \(A\)-module, we say that \(M\) is Cohen-Macaulay over \(A\) if \(M_{m}\) is Cohen-Macaulay over \(A_{m}\) for all maximal ideals \(m\) of \(A\) such that \(M_{m} \neq 0\) (in particular, the zero module is Cohen-Macaulay). We say that \(M\) is a maximal Cohen-Macaulay \(A\)-module if \(M_{m}\) is a maximal Cohen-Macaulay \(A_{m}\)-module for all maximal ideals \(m\) of \(A\). Finally we say that \(A\) is a Cohen-Macaulay ring if it is Cohen-Macaulay as an \(A\)-module.

Now let \(R = k[x_0, \ldots, x_N]\) as above, and denote by \(m_R\) the irrelevant maximal ideal \(R + \) of \(R\). Then a nonzero finitely generated graded \(R\)-module \(M\) is Cohen-Macaulay if and only if \(\text{depth}_{m_R} M = \dim M\), that is, if and only if \(m_R\) contains an \(M\)-regular sequence of length equal to \(\dim M\) (see [8], Proposition 1.5.15 and Exercise 2.1.27).

**Definition 1.2.2** Let \(X\) be a closed subscheme of \(\mathbb{P}^N\). If \(\dim X \geq 1\), we define \(A_X\) to be the ring \(H^0_*(\mathbb{P}^N, \mathcal{O}_X)\); if \(\dim X = 0\), we define \(A_X\) to be the homogeneous coordinate ring \(R_X\) of \(X\). We say that \(X\) is arithmetically Cohen-Macaulay (in short ACM) if the homogeneous coordinate ring \(R_X\) of \(X\) is a Cohen-Macaulay \(R\)-module. We say that \(X\) is quasi arithmetically Cohen-Macaulay (in short quasi ACM) if the ring \(A_X\) is a finitely generated Cohen-Macaulay \(R\)-module.

**Remark 1.2.3** An ACM subscheme \(X \subseteq \mathbb{P}^N\) is quasi ACM. This follows from the definition if \(\dim X = 0\). If \(\dim X \geq 1\), we look at the exact sequence:

\[
0 \to H^0_{m_\mathfrak{m}}(R_X) \to R_X \to A_X \to H^1_{m_\mathfrak{m}}(R_X) \to 0
\]

where \(H^j_{m_\mathfrak{m}}\) denotes the \(j^{th}\) local cohomology group with respect to the irrelevant maximal ideal \(m_\mathfrak{m}\) of \(R\). Since \(X\) is ACM of positive dimension, the ring \(R_X\) is Cohen-Macaulay of dimension at least 2. Hence the depth of \(R_X\) with respect to \(m_\mathfrak{m}\) is at least 2, and this implies:

\[
H^0_{m_\mathfrak{m}}(R_X) = H^1_{m_\mathfrak{m}}(R_X) = 0
\]

We conclude that \(A_X\) is isomorphic to \(R_X\), and hence that \(X\) is quasi ACM. More generally, we see that a quasi ACM subscheme of positive dimension
is ACM if and only if 
\[ H^1_{m_R}(R_X) \cong H^1_J(P^N, J_X) = 0. \]

Let \( L \subseteq P^N \) be a linear subspace of codimension \( t + 1 \). Associated to \( L \) there is a morphism 
\[ \phi_L : P^N \setminus L \to P^t \]
called projection from \( L \) and defined as follows on closed points: \( P^t \) is the projective space consisting of linear subspaces of \( P^N \) having codimension \( t \) and containing \( L \), and \( \phi_L \) maps a point \( P \in P^N \setminus L \) to the linear span of \( P \) and \( L \). If \( S_L \) denotes as above the polynomial subring of \( R \) generated by the linear equations of \( L \), the projection \( \phi_L \) corresponds to the injective map of graded rings \( S_L \to R \).

**Proposition 1.2.4** Let \( X \) be a closed subscheme of \( P^N \). If \( X \) has dimension \( t \geq 1 \), the following conditions are equivalent:

1. \( X \) is equidimensional and \( A_X \) is a Noetherian Cohen-Macaulay ring.
2. \( X \) is quasi ACM.
3. We have:
   \[ H^i(X, O_X) = 0 \quad \text{for} \quad 1 \leq i \leq t - 1 \]
   \[ H^0(X, O_X(n)) = 0 \quad \text{for} \quad n \ll 0 \]
4. \( A_X \) is a free finitely generated graded \( S_L \)-module for some (resp. every) linear subspace \( L \subseteq P^N \) of codimension \( t + 1 \) which does not meet \( X \).

Furthermore, if \( X \) has dimension zero, \( X \) is ACM and satisfies conditions 1, 2 and 4 above.

**Proof** First of all note that since \( O_X \) is a coherent sheaf of \( O_{P^N} \)-modules, \( A_X \) is finitely generated over \( R \) if and only if \( H^0(X, O_X(n)) = 0 \) for \( n \ll 0 \), hence \( A_X \) is Noetherian if and only if it is finitely generated over \( R \). Now, since \( R \to A_X \) is a finite map of rings, \( A_X \) is Cohen-Macaulay as an \( R \)-module if and only if it is Cohen-Macaulay as an \( A_X \)-module and for all homogeneous maximal ideals \( m \) of \( A_X \) we have \( \dim (A_X)_m = \dim_R A_X \). The latter condition means that \( X \) is equidimensional. Therefore 1 and 2 are equivalent. Next assume that condition 2 holds. Then \( A_X \) is finitely generated over \( R \) hence 
\[ H^0(X, O_X(n)) = 0 \quad \text{for} \quad n \ll 0 \]
and
\[ \dim A_X = t + 1 \]
If \( m_R \) denotes the irrelevant maximal ideal of \( R \), we have isomorphisms
\[ H^i_*(X, O_X) \cong H^{i+1}_{m_R}(A_X) \]
for \( i \geq 1 \). Since \( A_X \) is Cohen-Macaulay of dimension \( t + 1 \), \( H^j_{m_R}(A_X) = 0 \) for \( j \leq t \). Hence
\[ H^i_*(X, O_X) = 0 \text{ for } 1 \leq i \leq t - 1 \]
Thus condition 2 implies condition 3. Now assume 3 holds and let \( L \) be a linear subspace of codimension \( t + 1 \) which does not meet \( X \). Consider the projection from \( L \):
\[ \phi : X \to \mathbb{P}^t \]
Since \( X \) does not meet \( L \), \( \phi \) is well defined and has finite fibers. By Stein’s factorization \( \phi \) is a finite morphism, hence if we set \( \mathcal{G} = \phi_* O_X \), \( \mathcal{G} \) is a coherent sheaf on \( \mathbb{P}^t \) and there are isomorphisms
\[ H^i(\mathbb{P}^t, \mathcal{G}(n)) \cong H^i(X, O_X(n)) \]
In particular,
\[ A_X \cong H^0_*(\mathbb{P}^t, \mathcal{G}) \]
Now using local cohomology with respect to \( m_S \) and reversing the argument above we see that \( A_X \) is a finitely generated \( S \)-module of depth \( t + 1 \). Since \( S \) has dimension \( t + 1 \), the graded version of the Auslander-Buchsbaum Theorem tells us that \( A_X \) is projective over \( S \), and using Nakayama’s Lemma we see that \( A_X \) is in fact a free finitely generated \( S \)-module. Thus 3 implies 4.
Next assume that 4 holds for some \( L \) not meeting \( X \). Then the map \( S_L \to A_X \) is finite and flat, hence \( A_X \) is Cohen-Macaulay and equidimensional, so 1 holds.
Finally assume that \( X \) has dimension zero. Then by definition \( A_X = R/I_X \) where \( I_X \) is a saturated homogeneous ideal, which means that \( m_R \) is not an embedded prime ideal of \( A_X \). Since \( \dim A_X = 1 \), this implies that \( A_X \) is a Cohen-Macaulay \( R \)-module, so that 2 holds. Arguing as above we see that 2 is in fact equivalent to 1 and 4 and we are done.

**Remark 1.2.5** It follows that the property of being quasi ACM depends on \( X \) and on the very ample sheaf \( O_X(1) \), but not on the linear system which defines the embedding in \( \mathbb{P}^N \).

**Corollary 1.2.6** Let \( X \) be a quasi ACM closed subscheme of \( \mathbb{P}^N \). Then \( X \) is locally Cohen-Macaulay and equidimensional.
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**Proof** If $\dim X = t$, by Proposition 1.2.4 there is a finite flat morphism $\phi : X \to \mathbb{P}^t$, hence $X$ is locally Cohen-Macaulay and every component of $X$ has dimension $t$.

**Corollary 1.2.7** Let $X \subseteq \mathbb{P}^N$ be quasi ACM of dimension at least two, and let $H$ be a hyperplane of equation $h$ which meets $X$ properly. Then $Y = X \cap H$ is quasi ACM and

$$A_Y \cong A_X/hA_X.$$ 

**Proof** Since $H$ meets $X$ properly and $X$ is locally Cohen-Macaulay, $h$ is not a zero divisor for $O_X$ and we have an exact sequence:

$$0 \to O_X(-1) \xrightarrow{h} O_X \to O_Y \to 0$$

Now use condition 3 in Proposition 1.2.4.

**Corollary 1.2.8** Let $X$ be a closed subscheme of $\mathbb{P}^N$ of dimension one. The following conditions are equivalent:

1. $X$ is quasi ACM.
2. $X$ is locally Cohen-Macaulay and equidimensional.
3. Every associated point of $X$ has dimension one.

**Proof** If $X$ is quasi ACM, by Corollary 1.2.6 $X$ is locally Cohen-Macaulay and equidimensional, and this implies that every associated point of $X$ has dimension one.

Suppose now that every associated point of $X$ has dimension one. By Proposition 1.2.4 to show that $X$ is quasi ACM it is enough to prove that

$$H^0(\mathbb{P}^N, O_X(n)) = 0 \text{ for } n \ll 0,$$

and this follows from lemma 1.2.1.

**Definition 1.2.9** A curve is a locally Cohen Macaulay equidimensional projective scheme of dimension one.

Note that Proposition 1.2.8 says that a closed subscheme of $\mathbb{P}^N$ of dimension one is a curve if and only if it is quasi ACM.

**Example 1.2.10** A closed subscheme of $\mathbb{P}^3$ is quasi ACM if and only if it is one of the following: a zero dimensional subscheme, a curve, or a surface (by which we mean the scheme of zeroes of some form).
Example 1.2.11 We can construct subschemes $X$ of arbitrary dimension which are quasi ACM, but not ACM, by projecting ACM subschemes to a lower dimensional projective space. This way we obtain subschemes which are not linearly normal; however the class of quasi ACM subscheme is stable under biliaison (see [26] for the theory of biliaison), so we can bilink them up until we obtain linearly normal quasi ACM subschemes which are not ACM. For an explicit example, any surface in the biliaison class of the Veronese surface in $\mathbb{P}^4$ is quasi ACM, but not ACM.

1.3 The Canonical Module of a quasi ACM Subscheme

In this section we introduce the canonical module of a quasi ACM scheme $X \subseteq \mathbb{P}^N$. Our basic reference for this material is Section 3.6 of [8].

We first need to establish a few facts on the ring structure of $A_X$. We borrow the definition of a *local ring from [8] page 35. This notion is important because the theory of finitely generated graded modules over a Noetherian *local ring is perfectly analogous to that of finitely generated modules over a Noetherian local ring: see [8] especially sections 1.5 and 3.6.

Definition 1.3.1 Let $A$ be a graded ring. A graded ideal $m$ of $A$ is called *maximal if every graded ideal that properly contains $m$ equals $A$. The ring $A$ is called *local if it has a unique *maximal ideal $m$.

If $(A, m)$ is *local, then either $A/m$ is a field or $A/m \cong \mathbb{F}[t, t^{-1}]$ where $F$ is a field and $t$ is a homogeneous element of positive degree which is transcendental over $F$. In particular, $m$ is a prime ideal, and $m$ is maximal if and only if $A$ has no homogeneous unit of positive degree. Note that a positively graded $k$-algebra $A$ with $A_0 \cong k$ is *local with *maximal ideal $A_+ = \oplus_{n \geq 1} A_n$, which is in fact a maximal ideal and contains every proper homogeneous ideal of $A$.

Now suppose $X \subseteq \mathbb{P}^N$ is a closed subscheme. If $X$ has dimension zero, then $A_X = R/I_X$ is a positively graded $k$-algebra with $(A_X)_0 = k$, so $A_X$ is a *local ring with no homogeneous unit of positive degree. If $X$ has positive dimension, we will assume that $X$ has no associated points of dimension zero, so that $A_X = H^0(X, O_X(n))$ is finitely generated. If $X$ is reduced and connected, then $A_X$ is a positively graded $k$-algebra with $(A_X)_0 = k$, so $A_X$ is a *local ring with no homogeneous unit of positive degree. More generally, assume only that $X$ is connected. The nilradical $N$ of $A_X$ is a homogeneous ideal, and the quotient ring $B = A_X/N$ is a subalgebra of $A_{X, red}$, and therefore a positively graded $k$-algebra with $B_0 \cong k$. It
1.3. THE CANONICAL MODULE OF A QUASI ACM SUBSCHEME

follows that $A_X$ is *local with *maximal ideal $N + (A_X)_+$, which is in fact a maximal ideal containing all proper homogeneous ideal of $A_X$. For an arbitrary closed subscheme $X$ of positive dimension with no associated points of dimension zero, we have

$$A_X \cong \bigoplus A_i$$

where $A_i = A_{X_i}$ and the $X_i$'s are the connected components of $X$ with the induced scheme structure. Hence $A_X$ is a finite direct sum of *local rings (so we may call $A_X$ a *semi-local ring). If $n_i$ denotes the unique *maximal ideal of $A_i$, we let

$$m_i = n_i + \oplus_{l \neq i} A_l.$$ 

The ideals $m_i$'s are precisely the *maximal ideals of $A_X$; in fact every proper homogeneous ideal of $A_X$ is contained in at least one of the $m_i$'s. We have isomorphisms $(A_X)_{m_i} \cong (A_i)_{n_i}$ and $A_X/m_i \cong A_i/n_i \cong \mathfrak{k}$, from which we see that the $m_i$'s are maximal ideals.

**Definition 1.3.2** Let $X \subseteq \mathbb{P}^N$ be a quasi ACM subscheme of dimension $t$. We define $\omega_X \cong \mathcal{E}xt_{\mathbb{P}^N}^{N-t}(\mathcal{O}_X, \mathcal{O}_{\mathbb{P}^N}(-N-1))$ to be the Grothendieck dualizing sheaf of $X$. We define the canonical module of $X$ to be the graded $A_X$ module

$$\Omega_X = \mathcal{E}xt_{R}^{N-t}(A_X, R(-N-1)).$$

We will show later that if $X$ is quasi ACM of positive dimension, then $\Omega_X$ is isomorphic to $H^0(X, \omega_X)$.

**Proposition 1.3.3** Let $X \subseteq \mathbb{P}^N$ be a quasi ACM subscheme of dimension $t$, and let $L$ be a linear subspace of codimension $t+1$ which does not meet $X$. Then there is an isomorphism of $A_X$-modules

$$\Omega_X \cong \mathcal{H}om_{S_L}(A_X, S_L(-t-1)).$$

**Proof** Let $S = S_L$. The $A_X$-module structure on $\mathcal{H}om_S(A_X, S(-t-1))$ is defined by

$$(a\phi)(b) = \phi(ab)$$

for $a \in A_X$, $\phi \in \mathcal{H}om_S(A_X, S(-t-1))$ and $b \in A_X$.

We know that $A_X$ is a *local ring if $t = 0$; if $t \geq 1$, $A_X$ is a direct sum of finitely many *local $R$-algebras $A_i$'s, which correspond to the connected components of $X$, all of dimension $t+1$ and finitely generated as $S$-modules. It is then enough to prove the statement for the $A_i$'s:

$$\mathcal{E}xt_{R}^{N-t}(A_i, R(-N-1)) \cong \mathcal{H}om_S(A_i, S(-t-1)).$$
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This follows from Proposition 3.6.12 in [8] applied twice to the finite maps of local rings $S \rightarrow A_i$ and $R \rightarrow A_i$.

We now describe some of the properties of $\Omega_X$.

**Proposition 1.3.4** Let $X \subseteq \mathbb{P}^N$ be quasi ACM of dimension $t$ and let $\Omega$ be the canonical module of $X$. Then

1. $\Omega$ is a maximal Cohen-Macaulay $A_X$-module.
2. The natural map $A_X \rightarrow \text{Hom}_{A_X}(\Omega, \Omega)$ is an isomorphism; in particular $\Omega$ is a faithful $A_X$-module.
3. Over any of the rings $S$, $R$ and $A_X$, $\Omega$ and $A_X$ have the same associated primes.
4. $\Omega$ is a torsion free $A_X$-module, that is, every nonzero divisor of $A_X$ is a nonzero divisor of $\Omega$.

**Proof** The first statement follows for example from the fact that $\Omega \cong \text{Hom}_S(A_X, S(-t-1))$ is free, hence maximal Cohen-Macaulay over $S$. The second statement follows (localizing at the maximal ideals) from Theorem 3.3.10 in [8]. Since $\Omega$ is a faithful and maximal Cohen-Macaulay as an $A_X$ module, $\Omega$ and $A_X$ have the same associated primes over the ring $A_X$ and hence over $S$ and $R$ as well: the associated primes over $S$ (resp. $R$) of a finitely generated $A_X$-module $M$ are the pullback of the associated primes of $M$ over $A_X$. Finally, $\Omega$ is torsion free over $A_X$ because it has the same associated primes as $A_X$.

We are now going to find the relation between the canonical module and the dualizing sheaf of $X$. Given a graded ring $A$ and a graded $A$-module $M$, we denote by $\tilde{M}$ the associated sheaf on $\text{Proj} A$.

**Proposition 1.3.5** Let $Y \subseteq \mathbb{P}^N$ be an arbitrary closed subscheme of dimension $t \geq 1$ and fix a linear subspace $L$ of codimension $t + 1$ which does not meet $Y$. Let $S = S_L$ be the polynomial subring of $R$ generated by the linear forms vanishing on $L$.

Assume $M$ is a graded finitely generated $R$-module such that $\tilde{M} \cong \mathcal{O}_Y$. Then

$$H^0_s(Y, \omega_Y) \cong \text{Hom}_S(M, S(-t-1))$$

as $R$-modules, where the $R$-module structure on $\text{Hom}_S(M, S(-t-1))$ is defined by

$$(r\phi)(m) = \phi(rm)$$

for $r \in R$, $\phi \in \text{Hom}_S(M, S(-t-1))$ and $m \in M$. 

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**Proof** We project $Y$ from $L$ to the projective space $\mathbb{P}^t$ of codimension $t$ linear subspaces containing $L$. Note that $S \cong H^0_0(\mathcal{O}_{\mathbb{P}^t})$ and the projection corresponds to the map of rings $S \rightarrow R_Y$. Since $L$ doesn’t meet $Y$, this projection is a finite morphism $f : Y \rightarrow \mathbb{P}^t$. The sheafification of $M$ over $\mathbb{P}^t$ is $f_*\mathcal{O}_Y$, hence

$$\text{Hom}_{\mathbb{P}^t}(f_*\mathcal{O}_Y, \mathcal{O}_{\mathbb{P}^t}(n)) \cong \text{Hom}_S(M, S)^n$$

where the right hand-side denotes the set of homogeneous homomorphisms of degree $n$. Since $f$ is finite and $M$ is finitely generated over $R$, $M$ is finitely generated over $S$ and we have

$$\text{Hom}_S(M, S) \cong \bigoplus_{n \in \mathbb{Z}} \text{Hom}_S(M, S)^n.$$

On the other hand, since $f$ is finite, by [20], Exercises III 6.10 and 7.2, we have:

$$f_*\omega_Y \cong \text{Hom}_{\mathbb{P}^t}(f_*\mathcal{O}_Y, \omega_{\mathbb{P}^t}) \cong \text{Hom}_{\mathbb{P}^t}(f_*\mathcal{O}_Y, \mathcal{O}_{\mathbb{P}^t}(-t - 1))$$

Taking global sections and summing over $n$ we find

$$H^0_0(\mathbb{P}^t, f_*\omega_Y) \cong \text{Hom}_S(M, S(-t - 1)).$$

Furthermore, $f$ finite implies $H^0_0(\mathbb{P}^t, f_*\omega_Y) \cong H^0_0(Y, \omega_Y)$ and thus we obtain the desired isomorphism, at least as far as the $S$-structures of the two modules are concerned. This isomorphism preserves the $R$-modules structures as well, because, if we give $\text{Hom}_{\mathbb{P}^t}(f_*\mathcal{O}_Y, \omega_{\mathbb{P}^t})$ in the natural way (as in the statement) the structure of a $f_*\mathcal{O}_Y$-module, then

$$f_*\omega_Y \cong \text{Hom}_{\mathbb{P}^t}(f_*\mathcal{O}_Y, \omega_{\mathbb{P}^t})$$

is an isomorphism of $f_*\mathcal{O}_Y$-modules.

**Corollary 1.3.6** Let $X \subseteq \mathbb{P}^N$ be a quasi ACM subscheme of dimension $t \geq 1$ Then there is an isomorphism of $A_X$-modules

$$\Omega_X \cong H^0_0(\mathbb{P}^N, \omega_X)$$

1.4 The Spectrum of a quasi ACM Subscheme of $\mathbb{P}^N$

The spectrum of a quasi ACM subscheme $X \subseteq \mathbb{P}^N$ is a finite sequence of integers which carry as information the Hilbert function of the Cohen-Macaulay ring $A_X$; in the terminology of [47], the spectrum of $X$ is the $h$-vector of $A_X$. 
Definition 1.4.1 Given a numerical function \( f : \mathbb{Z} \to \mathbb{Z} \), we define its first difference function \( \partial f \) by the formula:

\[
\partial f(n) = f(n) - f(n - 1)
\]

and by induction we let

\[
\partial^r f = \partial \partial^{r-1} f \quad \text{for } r \geq 2
\]

Proposition 1.4.2 Let \( X \subseteq \mathbb{P}^N \) be a quasi ACM subscheme of dimension \( t \). Denote by

\[
a_n = \dim_k (A_X)_n
\]

the Hilbert function of \( A_X \).

There exists a unique sequence of nonnegative integers \( \{h_X(n)\} \) with the following properties:

1. For some (resp. every) linear subspace \( L \subseteq \mathbb{P}^N \) of codimension \( t + 1 \) which does not meet \( X \), we have an isomorphism

\[
A_X \cong \bigoplus_{n \in \mathbb{Z}} S_L(-n)^{h_X(n)}
\]

where \( S_L \) is the polynomial subring of \( R \) generated by the linear forms vanishing on \( L \).

2. We can express the Hilbert function in terms of \( \{h_X(n)\} \):

\[
a_n = \sum_{k \leq n} \binom{n - k + t}{t} h_X(k)
\]

3. Conversely, the Hilbert function determines the sequence \( \{h_X(n)\} \) via the formula

\[
h_X(n) = \partial t a_n
\]

Furthermore, only finitely many among the integers \( h_X(n) \) are nonzero.

Proof Uniqueness is obvious because of 3. We now show existence. Fix a linear subspace \( L \subseteq \mathbb{P}^N \) of codimension \( t + 1 \) which does not meet \( X \). By Proposition 1.2.4 \( A_X \) is a free finitely generated graded \( S_L \)-module, hence we have

\[
A_X \cong \bigoplus_{n \in \mathbb{Z}} S_L(-n)^{h_X(n)}
\]
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for some sequence of nonnegative integers $\{h_X(n)\}$ satisfying $h_X(n) = 0$ except for finitely many values of $n$.

Since

$$\dim(S_L)_n = \begin{cases} \binom{n+t}{t} & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

we find that

$$a_n = \sum_{k \leq n} \binom{n-k+t}{t} h_X(k).$$

Differentiating this we obtain

$$h_X(n) = \partial^{t+1} a_n$$

and we are done.

The sequence $\{h_X(n)\}$ is essentially what we will call the spectrum of $X$. According to the terminology of [48] we should call it the $h$-vector of the Cohen-Macaulay ring $A_X$. For convenience of notation, we prefer to think the spectrum as a set of integers with multiplicities, in which the integer $n$ occurs $h_X(n)$ times. We will use the notation $\{n^{b(n)}\}$ for the set of integers with multiplicities $B$ in which $n$ occurs $b(n)$ times. If $B$ is finite, we will also write

$$B = \{n_1^{b(n_1)}, \ldots, n_r^{b(n_r)}\}$$

where $n_1, \ldots, n_r$ are the integers for which $b(n) > 0$, that is, the integers which occur as elements of $B$. Finally we suppress the ”exponent” $b(n)$ if $b(n) = 1$. For example, we write $\{0, 1^2, 2^3\}$ for the set $\{0, 1, 1, 2, 2, 2\}$.

**Definition 1.4.3** Let $X \subseteq \mathbb{P}^N$ be a quasi ACM subscheme and let $h_X$ be the sequence of integers associated to $X$ as in Proposition 1.4.2. The set of integers with multiplicities $\{n^{h_X(n)}\}$ is called the spectrum of $X$ and is denoted by the symbol $sp_X$.

**Example 1.4.4** The spectrum of a linear subspace $M$ is $sp_M = \{0\}$. This by our convention means that $h_M(0) = 1$ and that $h_M(n) = 0$ for any nonzero integer $n$.

**Example 1.4.5** If $X$ is connected and reduced or if $X$ is arithmetically Cohen-Macaulay, then 0 occurs exactly once in $sp_X$ and every other integer
in the spectrum is positive. Indeed, under these assumptions \( \dim(A_X)_n \)
equals zero for \( n < 0 \) and one for \( n = 0 \). Similarly, if \( X \) is reduced, every
integer in \( sp_X \) is nonnegative.

**Example 1.4.6** On the other hand, the spectrum of a nonreduced scheme
may contain negative integers. The simplest example is given by double
structures on a line in \( \mathbb{P}^3 \). We write \( R = k[x, y, z, w] \) for the coordinate
ring of \( \mathbb{P}^3 \). Let \( C \) be a double structure on the line \( z = w = 0 \) with
arithmetic genus \( -r \leq 0 \). The homogeneous ideal of \( C \) is generated by
\( z^2, zw, w^2, zf + wg \) where \( f, g \in k[x, y] \) are homogeneous polynomials of
degree \( r \), forming a regular sequence if \( r > 0 \) (and not both zero if \( r = 0 \)).
One can check that \( A_C \) is a free graded \( k[x, y] \)-module with generators 1
and \( z/g = -w/f \). Hence \( sp_C = \{0, 1 - r\} \).

**Example 1.4.7** If \( X \) has positive dimension and \( X \) is the disjoint union
of \( Y \) and \( Z \), then the spectrum of \( X \) is the union of the spectra of \( Y \) and
\( Z \), that is,
\[
h_X(n) = h_Y(n) + h_Z(n).
\]
This follows from the fact that \( A_X \) is the direct sum of \( A_Y \) and \( A_Z \). In
particular, the spectrum of \( X \) is the union of the spectra of the connected
components of \( X \). Note that this is false for a zero dimensional scheme.
For example, it follows easily from the definition that the spectrum of one
point with the reduced structure is \( \{0\} \), the spectrum of two reduced points
is \( \{0, 1\} \), the spectrum of three reduced points is \( \{0, 1, 2\} \) if the points are
not collinear, and \( \{0, 1, 2\} \) if the points are collinear.

**Proposition 1.4.8** Let \( X \) be a quasi ACM subscheme of \( \mathbb{P}^N \) of dimension
t with spectrum \( sp_X = \{n^{h_X(n)}\} \). Then:

1. The Hilbert polynomial \( P_X(n) \) is given by the formula:
\[
P_X(n) = \sum_{k \in \mathbb{Z}} \binom{n-k+t}{t} h_X(k)
\]
In particular
\[
\deg X = \sum_{n \in \mathbb{Z}} h_X(n),
\]
while for the arithmetic genus \( g(X) \) we have:
\[
(-1)^t g(X) = \sum_{k \in \mathbb{Z}} \binom{t-k}{t} h_X(k) - 1
\]
2. If \( t \geq 1 \), we have
\[
h^t(X, \mathcal{O}_X(n)) = \sum_{k \geq n+t+1} \binom{k-1-n}{t} h_X(k).
\]

**Proof** For \( n \gg 0 \) we have by Proposition 1.4.2
\[
\dim(A_X)_n = \sum_{k \in \mathbb{Z}} \binom{n-k+t}{t} h_X(k).
\]
Since the right hand side is a polynomial in \( n \) and \( \dim(A_X)_n = P_X(n) \) for \( n \gg 0 \), we see that the right hand side is the Hilbert polynomial of \( X \). Its leading term is
\[
\sum_{k \in \mathbb{Z}} h_X(k) \frac{n^t}{t!}.
\]
therefore \( \deg X = \sum h_X(n) \). By definition of the arithmetic genus, we have
\[
(-1)^t g(X) = P_X(0) - 1 = \sum_{k \in \mathbb{Z}} \binom{t-k}{t} h_X(k) - 1.
\]
Now assume that \( t \geq 1 \). Then
\[
P_X(n) = h^0(X, \mathcal{O}_X(n)) + (-1)^t h^t(X, \mathcal{O}_X(n))
\]
hence
\[
(-1)^t h^t(X, \mathcal{O}_X(n)) = \sum_{k \geq n+1} \binom{n-k+t}{t} h_X(k).
\]
Our claim now follows from the fact that the binomial coefficient
\[
\binom{n-k+t}{t} = \frac{(n-k+t) \cdot (n-k+t-1) \cdots (n-k+1)}{t!}
\]
equals \((-1)^t \binom{k-1-n}{t}\) and is zero for \( n + 1 \leq k \leq n + t \).

**Proposition 1.4.9** Let \( X \subseteq \mathbb{P}^N \) be a quasi ACM subscheme of dimension \( t \) with spectrum \( \{n^{h_X(n)}\} \). If \( L \subseteq \mathbb{P}^N \) is a linear subspace of codimension \( t+1 \) which does not meet \( X \), we have an isomorphism:
\[
\Omega_X \cong \bigoplus_{n \in \mathbb{Z}} S_L(-n)^{h_X(t+1-n)}.
\]
In particular, if \( \dim X \geq 1 \), then
\[
h^0(X, \omega_X(n)) = \sum_{k \leq n} \binom{n-k+t}{t} h_X(t+1-k).
\]
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**Proof** We have $A_X \cong \bigoplus_{n \in \mathbb{Z}} S_L(-n)^{h_X(n)}$ by Proposition 1.4.2, so the desired isomorphism follows from Proposition 1.3.3 which says that

$$\Omega_X \cong \text{Hom}_{S_L}(A_X, S_L(-t-1)).$$

The formula for $h^0(X, \omega_X(n))$ follows from Proposition 1.3.6 (or by Serre’s duality from Proposition 1.4.8).

**Definition 1.4.10** We define the *index of speciality* $e(X)$ of a quasi ACM subscheme $X \subseteq \mathbb{P}^N$ to be the largest integer $n$ such that $\Omega_X$ is nonzero in degree $-n$.

**Remark 1.4.11** If $X$ has positive dimension, this reduces to the usual definition: $e(X)$ is the largest integer $n$ such that $H^0(X, \omega_X(-n)) \neq 0$.

**Corollary 1.4.12** Let $X$ be a quasi ACM subscheme of $\mathbb{P}^N$ of dimension $t$ with spectrum $sp_X = \{n^{h_X(n)}\}$ and index of speciality $e = e(X)$. Then

$$h_X(e + t + 1) = \dim(\Omega_X) - e > 0$$

and

$$e + t + 1 = \max\{n : h_X(n) > 0\}$$

**Proof** The statement is an immediate consequence of Proposition 1.4.9.

**Definition 1.4.13** Let $X$ be a quasi ACM subscheme of $\mathbb{P}^N$ and let $l \in \mathbb{Z}$ be an integer. We say that $X$ is subcanonical of index $l$ if there exists an isomorphism $\Omega_X \cong A_X(l)$ (if $\dim X \geq 1$, this is equivalent to $\omega_X \cong \mathcal{O}_X(l)$). We say that $X$ is canonical if it is subcanonical of index one.

The spectrum of subcanonical curves is symmetric:

**Proposition 1.4.14** Let $X$ be a quasi ACM subscheme of $\mathbb{P}^N$ of dimension $t$ with spectrum $sp_X = \{n^{h_X(n)}\}$. If $X$ is subcanonical of index $l$, then the spectrum of $X$ has following symmetry:

$$h_X(n) \cong h_X(t + 1 + l - n)$$

for all integers $n$.

**Proof** By Proposition 1.4.9 we have

$$\Omega_X \cong \bigoplus_{n \in \mathbb{Z}} S_L(-n)^{h_X(t+1-n)}$$

while

$$A_X \cong \bigoplus_{n \in \mathbb{Z}} S_L(-n)^{h_X(n)}$$

by Proposition 1.4.2. Now the statement is a consequence of the isomorphism $\Omega_X \cong A_X(l)$. 

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**Definition 1.4.15** Let $X \subseteq \mathbb{P}^N$ be a quasi ACM subscheme of dimension $t$. We define the Rao module of $X$ to be the graded $R$-module

$$M_X = A_X / R_X.$$ 

**Remark 1.4.16** Note that $M_X$ is a module of finite length and that a quasi ACM subscheme $X$ is ACM if and only if $M_X = 0$. Furthermore, if $X$ has dimension $\geq 1$, then

$$M_X = H^1_*(\mathbb{P}^N, J_X).$$

The following Proposition reduces the study of the spectrum of an arbitrary quasi ACM (resp. ACM) subscheme to the one dimensional case (resp. the zero dimensional case).

**Proposition 1.4.17** Let $X \subseteq \mathbb{P}^N$ be a quasi ACM subscheme of dimension $t \geq 1$, let $H$ be a hyperplane meeting $X$ properly, and let $Y$ be the scheme theoretic intersection $X \cap H$. If $\dim X \geq 2$, then $X$ and $Y$ have the same spectrum. If $\dim X = 1$, let $M = M_X$ be the Rao module of $X$ and define

$$q(n) = \dim(M/hM)_n$$

where $h$ is an equation of $H$. Then

$$h_X(n) = h_Y(n) + \partial q(n)$$

**Proof** If $\dim X \geq 2$, the statement follows from Corollary 1.2.7. So assume $\dim X = 1$: then $Y$ has dimension zero, hence it is ACM. Let $A = A_X$ and

$$Q = \text{coker}(h : M(-1) \to M).$$

We look at the commutative diagram with exact rows:

$$
\begin{array}{ccccccc}
0 & \to & R_X(-1) & \to & A(-1) & \to & M(-1) & \to & 0 \\
& & \downarrow h & & \downarrow h & & \downarrow h & \\
0 & \to & R_X & \to & A & \to & M & \to & 0
\end{array}
$$

The first two vertical maps are injective because $H$ meets $X$ properly; hence, if we set

$$N = \ker(h : M(-1) \to M)$$

we obtain an exact sequence:

$$0 \to N \to R_X/hR_X \to A/hA \to Q \to 0.$$
Note that $M$ has finite length, so $N$ and $Q$ have finite length as well. We break the above sequence into two short exact sequences:

(1.1) \[ 0 \rightarrow N \rightarrow R_X/hR_X \rightarrow R/J \rightarrow 0 \]
(1.2) \[ 0 \rightarrow R/J \rightarrow A/xA \rightarrow Q \rightarrow 0 \]

where $J$ is some homogeneous ideal in $R$. Since $N$ has finite length and the sheaf associated to $R_X$ is $O_X$, the first sequence tells us that that sheaf associated to $R/J$ is $O_Y$, while the second one implies that $R/J$ has depth at least one since $\text{depth} A/hA \geq 1$. Hence $J$ is a saturated ideal and therefore it is equal to $I_Y$, that is, $R/J = R_Y$. Thus (1.2) becomes:

(1.3) \[ 0 \rightarrow R_Y \rightarrow A/hA \rightarrow Q \rightarrow 0 \]

from which we see that

(1.4) \[ \dim A_n - \dim A_{n-1} = \dim(R_Y)_n + q(n) \]

Now recall that

\[
\begin{align*}
    h_X(n) &= \partial^2(\dim A_n) \\
    h_Y(n) &= \partial(\dim(R_Y)_n)
\end{align*}
\]

Thus if we take first differences in equation (1.4) we obtain

\[
    h_X(n) = h_Y(n) + \partial q(n)
\]

and we are done.

### 1.5 The Spectrum of a Complete Intersection

To give a first important example, we compute the spectrum of a complete intersection. Modulo the terminology this is of course well known: see for example Exercise 4.3.14 in [8]. We will use the following notations: given a sequence $a = \{a_1, \ldots, a_r\}$ of $r \geq 1$ positive integers we consider the polynomial

\[
P_a(t) = \prod_{j=1}^{r} (1 + t + \cdots + t^{a_j-1})
\]

and we let $\epsilon_a(n)$ be the coefficient of $t^n$ in $P_a$. It is clear that $\epsilon_a(n)$ is a nonnegative integer equal to the cardinality of the set:

\[
    E_a(n) = \{(i_1, \ldots, i_r) : 0 \leq i_k \leq a_k - 1 \text{ and } \sum_{k=1}^{r} i_k = n\}
\]

The main properties of the integers $\epsilon_a(n)$ are given by the following Lemma.
1.5. THE SPECTRUM OF A COMPLETE INTERSECTION

Lemma 1.5.1 Let $a = \{a_1, \ldots, a_r\}$ be sequence of $r \geq 1$ positive integers and let $\delta = \deg P_a = \sum_{j=1}^{r}(a_j - 1)$. The integers $\epsilon(n) = \epsilon_a(n)$ have the following properties:

1. Symmetry: $\epsilon(n) = \epsilon(\delta - n)$

2. $\epsilon(0) = \epsilon(\delta) = 1$ and $\epsilon(n) = 0$ for all $n < 0$ and all $n > \delta$.

3. For $r \geq 2$ we have

   $$\partial \epsilon_{a_1, \ldots, a_r}(n) = \epsilon_{a_1, \ldots, a_r-1}(n) - \epsilon_{a_1, \ldots, a_r-1}(n - a_r)$$

4. The sequence $\{\epsilon(n)\}$ is nondecreasing to the left of $\delta/2$ and nonincreasing to the right of $\delta/2$.

5. $\sum_{n} \epsilon(n) = a_1 \times \cdots \times a_r$

Proof The symmetry property follows from the fact that mapping $(i_1, \ldots, i_r)$ to $(a_1 - 1 - i_1, \ldots, a_r - 1 - i_r)$ we obtain a bijection $E_a(n) \to E_a(\delta - n)$.

Property 2 follows immediately from the definition. To prove 3, if we let $b = \{a_1, \ldots, a_{r-1}\}$, we have to show that

$$\partial \epsilon_a(n) = \epsilon_b(n) - \epsilon_b(n - a_r)$$

This is in fact equivalent to:

$$\epsilon_a = \sum_{k=0}^{a_r-1} \epsilon_b(n-k)$$

which follows from the equalities

$$P_a(t) = P_b(t) \cdot (1 + \cdots + t^{a_r-1}) = \sum_{n} \left( \sum_{k=0}^{a_r-1} \epsilon_b(n-k) \right) t^n$$

To prove 4, by symmetry it is enough to show that $\epsilon(n) - \epsilon(n - 1) \geq 0$ for $n \leq \delta/2$. We prove this by induction on $r$. For $r = 1$ it is clear since $\epsilon_a(n) = 1$ for $0 \leq \delta_a = a_1 - 1$. Now we assume that 4 holds for $b = \{a_1, \ldots, a_{r-1}\}$. Note that $\delta_b = \delta_a - a_r + 1$. We fix $n \leq \delta_a/2$. We have

$$2n - a_r \leq \delta_a - a_r = \delta_b - 1$$

hence $n - a_r < \delta_b/2$. By the induction hypothesis we have

$$\epsilon_b(n - a_r) \leq \epsilon_b(m)$$
for \( n-a_r \leq m \leq \delta_b/2 \) and hence by symmetry for \( n-a_r \leq m \leq \delta_b-(n-a_r) \).

Now \( 2n \leq \delta_a \) implies 
\[
n \leq \delta_b - (n-a_r)
\]
hence we can take \( m = n \) and conclude that 
\[
\partial \epsilon_a(n) = \epsilon_b(n) - \epsilon_b(n-a_r) \geq 0
\]
which proves 4.

Finally 5 follows from the fact that \( \sum_n \epsilon(n) \) is equal to the cardinality of the set 
\[
E = \{(i_1, \ldots, i_r) : 0 \leq i_k \leq a_k - 1\}
\]
which is equal to the product of the \( a_j \)'s.

**Proposition 1.5.2** Let \( X \subseteq \mathbb{P}^N \) be the complete intersection of \( r \) hypersurfaces of degrees \( a_1, \ldots, a_r \) which meet properly. Then \( X \) is ACM and the spectrum of \( X \) is given by the formula:
\[
h_X(n) = \epsilon_{a_1, \ldots, a_r}(n) \quad \text{for all } n \in \mathbb{Z}.
\]

**Proof** \( X \) is ACM because \( R_X \) is a quotient of \( R \) by a regular sequence of homogeneous elements. We let \( b_X(n) = \dim(R_X)_n \) denote the Hilbert function of \( R_X \). Since \( X \) has dimension \( N-r \), we need to prove that 
\[
\partial N+1-r b_X(n) = \epsilon_{a_1, \ldots, a_r}(n)
\]
We prove this by induction on \( r \). It is convenient to start the induction with \( r = 0 \), in which case we take \( X = \mathbb{P}^N \) and we define for the empty sequence \( \epsilon_\emptyset(n) \) to be zero for all \( n \neq 0 \) and one for \( n = 0 \). This is consistent with our previous definitions because
\[
\partial \epsilon_{a_1}(n) = \epsilon_\emptyset(n) - \epsilon_\emptyset(n-a_1).
\]
Our statement for \( r = 0 \) becomes
\[
\partial N+1 b_{\mathbb{P}^N} = \epsilon_\emptyset(n)
\]
which is easily seen to hold. To proceed by induction, we assume that the Proposition holds for \( Y \), the complete intersection of the first \( r-1 \) hypersurfaces cutting out \( X \). We have an exact sequence:
\[
0 \to R_Y(-a_r) \to R_Y \to R_X \to 0
\]
from which we see that \( b_X(n) = b_Y(n) - b_Y(n-a_r) \). Using the induction hypothesis we see that
\[
\partial \partial^{N+1-r} b_X(n) = \partial \partial^{N+1-(r-1)} [b_Y(n) - b_Y(n-a_r)] = \epsilon_{a_1,\ldots,a_{r-1}}(n) - \epsilon_{a_1,\ldots,a_{r-1}}(n-a_r)
\]
By Lemma 1.5.1 this gives
\[
\partial \partial^{N+1-r} b_X(n) = \partial \epsilon_{a_1,\ldots,a_r}(n)
\]
Since both \( b_X(n) \) and \( \epsilon_{a_1,\ldots,a_r}(n) \) are zero for \( n \ll 0 \), we deduce that
\[
\partial^{N+1-r} b_X(n) = \epsilon_{a_1,\ldots,a_r}(n)
\]
and we are done.

**Corollary 1.5.3** If \( F \subseteq \mathbb{P}^N \) is a hypersurface of degree \( d \), then
\[
sp_F = \{0, 1, \ldots, d-1\}.
\]
If \( F = F_{a,b} \) is the complete intersection of two hypersurfaces of degrees \( a \) and \( b \) with \( a \leq b \), then
\[
sp_F = \{0, 1^2, \ldots, (a-1)^a, \ldots, (b-1)^a, b^{a-1}, \ldots, (a+b-3)^2, a+b-2\}
\]
that is
\[
h_F(n) = \begin{cases} n+1 & \text{if } 0 \leq n \leq a-1 \\ a & \text{if } a \leq n \leq b-1 \\ a+b-1-n & \text{if } b \leq n \leq a+b-2 \end{cases}
\]
while \( h_F(n) = 0 \) for all other values of \( n \).

**Remark 1.5.4** For a complete intersection \( X \subseteq \mathbb{P}^N \) of \( r \) hypersurfaces of degrees \( a_1, \ldots, a_r \) we have
\[
\Omega_X \cong A_X(\sum a_j - N - 1)
\]
while the dimension of \( X \) is \( N-r \). It follows from Proposition 1.4.14 that
\[
h_X(n) = h_X(\sum (a_j - 1) - n).
\]
This gives another proof of 1 in Lemma 1.5.1.
1.6 Macaulay’s Theorem and The Spectrum of an ACM Subscheme

In this section we describe the sequences of integers which occur as spectra of ACM subschemes of \( \mathbb{P}^N \): this is an easy Corollary of a Theorem of Macaulay which determines the possible Hilbert functions of homogeneous \( k \)-algebras. We learned these results from [8] Sections 4.2 and 4.3, which are based on R. Stanley’s paper [47] but also include M. Green’s proof of Macaulay’s Theorem. We then examine the case of codimension two subschemes in more details.

We first reproduce (without proof) Lemma 4.2.6 and the definition of the Macaulay representation of a nonnegative integer from [8].

**Lemma 1.6.1** Let \( d \) be a positive integer. Any nonnegative integer \( a \) can be written uniquely in the form

\[
a = \binom{k(d)}{d} + \binom{k(d-1)}{d-1} + \cdots + \binom{k(1)}{1},
\]

where the \( k(i) \)'s are integers satisfying \( k(d) > k(d-1) > \cdots > k(1) \geq 0 \).

**Definition 1.6.2** Let \( d \) be a positive integer and let \( a \) be a nonnegative integer. The sum in the Lemma is called the \( d \)-th Macaulay representation of \( a \), and \( k(d), \ldots, k(1) \) are called the \( d \)-th Macaulay coefficients of \( a \). We define

\[
a^{(d)} = \binom{k(d) + 1}{d+1} + \binom{k(d-1) + 1}{d} + \cdots + \binom{k(1) + 1}{2}
\]

The following Lemma may help to get familiar with these definitions:

**Lemma 1.6.3** Let \( d \) and \( k \) be positive integers and let \( a \) be a nonnegative integer. Then

\[
a^{(d)} = a \quad \text{for} \quad d \geq a
\]

\[
\binom{d+k}{k}^{(d)} = \binom{d+k+1}{k}
\]

In particular, we have \( 0^{(d)} = 0 \), \( 1^{(d)} = 1 \) and \( (d+1)^{(d)} = d+2 \) for all \( d \).
1.6. MACAULAY’S THEOREM AND THE SPECTRUM OF AN ACM SUBSCHEME

Proof. If \( d \geq a \), then
\[
a = \binom{d}{d} + \binom{d-a+1}{d-a+1} + \ldots + \binom{d-a+1}{d-a+1}
\]
hence the \( d \)-th Macaulay coefficients of \( a \) are \( d, d-1, \ldots, d-a+1, d-a-1, \ldots, 0 \) if \( a \geq 1 \), and \( d-1, d-2, \ldots, 0 \) if \( a = 0 \). It follows that
\[
a^{(d)} = a \text{ for } d \geq a.
\]
The equality
\[
\binom{d+k}{k}^{(d)} = \binom{d+k+1}{k}
\]
follows from the definition and the formula
\[
\binom{d+k}{k} = \binom{d+k}{d}.
\]
Taking \( k = 1 \) we obtain \( (d+1)^{(d)} = d+2 \).

We can now state Macaulay’s Theorem. For a proof (due to M. Green) see for example [8], Theorem 4.2.10.

Theorem 1.6.4 (Macaulay). Let \( K \) be a field, let \( h : \mathbb{N} \to \mathbb{N} \) be a numerical function and let \( m = h(1) \). The following conditions are equivalent:

1. there exists a proper homogeneous ideal \( I \) in \( R = K[x_1, \ldots, x_m] \) such that \( h(n) \) is the Hilbert function of \( R/I \), that is,
\[
\dim_K (R/I)_n = h(n) \text{ for all } n \in \mathbb{N}
\]
2. we have \( h(0) = 1 \) and \( h(n+1) \leq h(n)^{(n)} \text{ for all } n \geq 1 \).

As a Corollary we obtain a characterization of the sets of integers which occur as spectra of ACM subschemes of \( \mathbb{P}^N \).

Corollary 1.6.5. Let \( h : \mathbb{N} \to \mathbb{N} \) be a sequence of nonnegative integers, only finitely many of which are nonzero, and let \( m = h(1) \). The following conditions are equivalent:

1. for all (resp. some) integers \( N \geq m \) there exists an ACM subscheme \( X \subseteq \mathbb{P}^N \) of codimension \( m \) with spectrum \( \{ n^{h(n)} \} \).
2. \( h(0) = 1 \) and \( h(n+1) \leq h(n)^{(n)} \text{ for all } n \geq 1 \).
Furthermore, in 1 we can take X to be reduced.

**Proof** Assume $X \subseteq \mathbb{P}^N$ is an ACM subscheme of codimension $m$ with spectrum $\{n^{h(n)}\}$. We choose a linear subspace $L$ of codimension $N - m + 1$ which does not meet $X$, and let $S = S_L$; then $R_X \otimes_S S/m_S$ is a quotient of the polynomial ring $R_{\mathbb{P}^N} \otimes_S S/m_S$ with Hilbert function $h(n)$. By Macaulay’s Theorem the sequence $\{h(n)\}$ must satisfy 2. Conversely, if 2 holds, by Macaulay’s Theorem there is a zero dimensional graded $k$-algebra $B = k[x_1, \ldots, x_m]/I$ with Hilbert function $h(n)$. Now, if $N \geq m$, define

$$X = \text{Proj} B[y_0, \ldots, y_{N-m}].$$

Since $B$ is zero dimensional, $B$ is Cohen-Macaulay, and so is $B[y_0, \ldots, y_{N-m}]$. Therefore $X$ is an ACM subscheme of $\mathbb{P}^N = \text{Proj} k[x_1, \ldots, x_m, y_0, \ldots, y_{N-m}]$ with the desired spectrum. The fact that we can take $X$ reduced follows from Proposition 4.3.1 in [8] if $N$ is sufficiently large; then we can use Bertini’s Theorem (we are working over an algebraically closed field $k$) to cut $N$ down to an arbitrary $N_1 \geq m$.

**Remark 1.6.6 ([47] page 61)** Assume $X \subseteq \mathbb{P}^N$ is ACM with spectrum $\{n^{h(n)}\}$. If for some $l \geq 1$ we have $h(l) \leq l$, then $h(n) \geq h(n+1)$ for all $n \geq l$. Indeed, by Lemma 1.6.3 we have:

$$a^{(d)} = a \text{ for } d \geq a$$

**Example 1.6.7** If $X$ is a hypersurface, it follows that there is an integer $l \geq 1$ such that $h_X(n) = 1$ for $0 \leq n < l$ and $h_X(n) = 0$ for $n \geq l$. We then see that $l = \deg X$. Of course, we already knew this.

The codimension two is the first nontrivial case of the theory, hence we study it in some detail.

Suppose $X \subseteq \mathbb{P}^N$ is ACM of codimension 2 with spectrum $\{n^{h(n)}\}$. Then $h(1)$ can be 0, 1 or 2. If it is 0 or 1, then $X$ is a hypersurface in some hyperplane $H$ and its spectrum is $\{0, 1, \ldots, \deg X - 1\}$. If $h(1) = 2$, then $h(2) \leq 2^{(1)} = 3$ since by Lemma 1.6.3 we have $(n+1)^{(n)} = n+2$. Continuing by induction we see that there exists $s \geq 1$ such that $h(n) = n + 1$ for $0 \leq n < s$ and $h(n) \geq h(n+1)$ for $n \geq s - 1$. This motivates the following definition:

**Definition 1.6.8** Given an integer $s \geq 1$, an $s$-sequence is a sequence of integers $h : \mathbb{N} \rightarrow \mathbb{N}$ satisfying the following properties:

$$h(n) = \begin{cases} 
  n + 1 & \text{if } 0 \leq n \leq s - 1 \\
  h(n + 1) & \text{if } s - 1 \leq n \\
  0 & \text{if } n \gg 0 \text{ or if } n < 0.
\end{cases}$$
1.6. MACAULAY’S THEOREM AND THE SPECTRUM OF AN ACM SUBSCHEME

Given an $s$-sequence $h$, we define $t = t(h)$ to be the least integer $n \geq s$ such that $h(t) < s = h(s - 1)$.

To recover the geometric meaning of this definition, we need another definition.

Definition 1.6.9 Let $Y$ be an arbitrary closed subscheme of $\mathbb{P}^N$. We define $s(Y)$ to be the smallest integer $n$ such that there is a hypersurface of degree $n$ containing $Y$. We define $t(Y)$ to be the smallest integer $m$ with the following property: there are two hypersurfaces $F$ and $G$ of degrees $s(Y)$ and $m$ respectively such that $F$ and $G$ contain $Y$, but $G$ does not contain $F$.

We can now restate Corollary 1.6.5 for ACM subschemes of codimension two:

Corollary 1.6.10 Let $h: \mathbb{N} \to \mathbb{N}$ be a sequence of nonnegative integers with $h(1) = m$, and let $t \geq s \geq 1$ be integers. The following conditions are equivalent:

1. for all (resp. some) integers $N \geq m$ there exists a (reduced) ACM subscheme $X \subseteq \mathbb{P}^N$ of codimension at most 2 such that $sp_X = \{n^{h(n)}\}$, $s(X) = s$ and $t(X) = t$.

2. $h$ is an $s$-sequence with $t(h) = t$.

We now give an independent proof of the implication 1 $\Rightarrow$ 2, together with some remarks we will need later. Let $T = k[z,w]$ be the polynomial ring in the variables $z$ and $w$ over the field $k$. Let $J \subseteq T$ be a homogeneous ideal such that $V = T/J$ has finite length and is nonzero. Equivalently, $V$ is a nonzero principal graded $T$-module of finite length. We define

- $h_V(n) = \dim(V_n)$ (the Hilbert function of $V$).
- $sp_V = \{n^{h_V(n)}\}$ (the spectrum of $V$).
- $s(V) = \inf\{n : J_n \neq 0\}$
- $t(V) = \inf\{n : \dim J_n > \dim T_{n-s}\}$
- $d(V) = \text{length}(V) = \sum h_V(n)$

Note that $1 \leq s(V) \leq t(V) < +\infty$ and that $t(V)$ is equal to the smallest integer $n$ such that there exist nonzero polynomials $f \in J_s$ and $g \in J_n$ with $g$ not a multiple of $f$. This notation is justified as follows: suppose that $X \subseteq \mathbb{P}^N$ is an ACM subscheme of codimension two, let $L$ be a line which
does not meet \( X \), and let \( S = S_L \). Since \( X \) is ACM, we have \( A_X = R_X \), and \( V = R_X \otimes_S k \) is a principal graded module over \( T \cong R \otimes_S k \) such that \( sp_V = sp_X \), \( s(V) = s(X) \) and \( t(V) = t(X) \).

**Proposition 1.6.11** Let \( T \) denote the polynomial ring in two variables over \( k \). Let \( V = T/J \) be a nonzero principal graded \( T \)-module of finite length. Let \( z \in T_1 \) be a general linear form. Then multiplication by \( z : V_n \to V_{n+1} \) is

1) injective for \( n \leq t(V) - 2 \).

2) surjective for \( n \geq s(V) - 1 \).

Furthermore, the Hilbert function \( h_V(n) \) is an \( s(V) \)-sequence with \( t(h_V) = t(V) \).

**Proof** Note that we are assuming that \( k \) is algebraically closed, but the proof we give requires only \( k \) to be infinite. Fix \( f \in J \) a nonzero homogeneous polynomial of minimal degree in \( J \). Then \( V \) is a quotient of \( T/f \) with \( V_n \cong (T/f)_n \) for \( n \leq t(V) - 1 \). Now, if \( z \in T_1 \) is general, \( z \) is not contained in any associated prime of \( T/f \), that is, \( z \) is \( T/f \)-regular. It follows that multiplication by \( z : (T/f)_n \to (T/f)_{n+1} \) is always injective and an isomorphism for \( n \geq s(V) - 1 \). This proves 1) and 2); the rest of the statement, which shows \( 1 \Rightarrow 2 \) in Corollary 1.6.10, now follows easily.

Next we are going to introduce the \( \gamma \)-character and the numerical character of \( V \). These characters are convenient ways of keeping track of the Hilbert function of \( V \). Our purpose here is to provide a dictionary between our notation and those of [32] and [15].

**Definition 1.6.12** Given a nonzero principal graded \( T \)-module \( V \) of finite length, we define the \( \gamma \)-character of \( V \) to be the numerical function

\[
\gamma_V = -\partial h_V.
\]

**Remark 1.6.13** If \( X \subseteq \mathbb{P}^N \) is ACM of codimension two, one can easily check that the gamma character \( \gamma_X \) of \( X \) as defined in [32] coincide with \( \gamma_V \), where \( V = R_X \otimes_S k \) as above.

In the terminology of [32], Proposition 1.6.11 says that \( \gamma_V \) is a positive character: we have

\[
\gamma_V(n) = \begin{cases} 
-1 & \text{if } 0 \leq n \leq s(V) - 1 \\
0 & \text{if } s(V) \leq n \leq t(V) - 1 \\
a_n & \text{if } n \geq t(V)
\end{cases}
\]

Furthermore, \( a_t(V) > 0 \) and \( \sum a_n = s(V) \).
1.7. PROPERTIES OF THE SPECTRUM

Definition 1.6.14 The numerical character of $V$ is the set of integers with multiplicities $n(V) = \{n^b_V(n)\}$ defined by the formula

$$b_V(n) = \begin{cases} 0 & \text{if } n \leq s(V) - 1 \\ \gamma_V(n) & \text{if } s(V) \leq n \end{cases}$$

Note that the definition makes sense by Proposition 1.6.11, that $t(V)$ is the smallest integer in $n(V)$ and that $n(V)$ consists of exactly $s(V)$ integers since

$$\sum_{n \geq s(V)} \gamma_V(n) = s(V).$$

Remark 1.6.15 If $X \subseteq \mathbb{P}^N$ is ACM of codimension two, the numerical character $n(X)$ of $X$ as defined by Gruson and Peskine coincide (modulo the notation) with $n(V)$, where $V = R_X \otimes S k$ as above.

Example If $V = k[z, w]/(z^2, w^2)$, we have $s(V) = t(V) = 2$, $sp_V = \{0, 1^2, 2\}$, $n(V) = \{2, 3\}$ and

$$\gamma_V(n) = \begin{cases} -1 & \text{if } n = 0, 1 \\ 1 & \text{if } n = 2, 3 \\ 0 & \text{otherwise} \end{cases}$$

If $X$ is a complete intersection of two quadric hypersurfaces, the spectrum, the gamma character and the numerical character of $X$ equal those of $V$.

1.7 Properties of the Spectrum

The main result in this section is Theorem 1.7.1 which describes some fundamental properties of the spectrum of a quasi ACM subscheme $X$ of $\mathbb{P}^N$. This theorem is modeled after corresponding results from the theory of the spectrum for torsion free sheaves on $\mathbb{P}^3$ (to which we will return in the following chapter); in particular, the hard part of the Theorem has its origin in [22], Proposition 5.1, and [42], Theorem 4.1 (which probably implies our statement when $X$ is a curve in $\mathbb{P}^3$).

Recall that the Rao module $M_X$ of $X$ was defined as $A_X/R_X$. We denote by $\mu_X(n)$ the number of minimal generators in degree $n$ of the Rao module over ring $R$, that is:

$$\mu_X(n) = \dim(M_X \otimes_R R/m^n_R)_n.$$ 

Theorem 1.7.1 Let $X \subseteq \mathbb{P}^N$ be a quasi ACM subscheme of dimension $t$ and index of speciality $e = e(X)$. Then $e + t + 1 \geq 0$ and the spectrum of $X$ has the following properties:
1. \( h_X(n) \geq 1 + \mu_X(n) \) for \( 0 \leq n \leq e(X) + t + 1 \).

2. Suppose there is an integer \( l \) satisfying \( 1 \leq l \leq e + t + 1 \) such that \( h_X(l) = 1 + \mu_X(l) \). Then

\[
h_X(n) = 1 \quad \text{and} \quad \mu_X(n) = 0 \quad \text{for} \quad l \leq n \leq e + t + 1.
\]

Furthermore, if \( l \leq e + t \), there is a linear subspace \( M \subseteq \mathbb{P}^N \) of dimension \( t + 1 \) such that \( X \) contains a hypersurface in \( M \) of degree \( e + t + 2 \).

3. \( h_X(n) = \mu_X(n) = 0 \) for \( n \geq e + t + 2 \).

Remark 1.7.2 When \( X \) is ACM, the Theorem is easy to prove, and its numerical statements follow from Macaulay’s Theorem and Corollary 1.4.12.

Remark 1.7.3 Statement 1 in the Theorem implies that the spectrum is connected in positive degrees: more precisely, the nonnegative integers appearing in the spectrum at least once are \( 0, 1, \ldots, e + t + 1 \). On the other hand, we have seen that the spectrum may have gaps in negative degrees. However, the theory of the spectrum for sheaves on \( \mathbb{P}^3 \) indicates that there should be some connectedness result on the left as well; as we don’t need any such result in the sequel, we have reserved this problem for the future.

Remark 1.7.4 The Theorem also says that for \( 1 \leq n \leq e + t \), then \( h_X(n) \geq 2 \) unless \( X \) has a component contained in a linear space of dimension \( t + 1 \). In case \( N = 3 \) and the characteristic is zero, we will later generalize this to a statement for curves with no component on a surface of degree \( s - 1 \) by a very different method.

To prove the Theorem we need the following Lemma:

Lemma 1.7.5 Let \( X \) be a quasi ACM subscheme of \( \mathbb{P}^N \) of dimension \( t \), let \( L \) be a linear subspace of codimension \( t + 1 \) which does not meet \( X \) and let \( S = S_L \). For a subset \( U \) of the set of minimal generators of \( A_X \) over \( S \) define \( \hat{U} \) to be the \( S \)-span of \( U \) in \( A_X \). Assume that:

- \( \hat{U} \) is an \( R \)-submodule of \( A_X \).
- \( \hat{U} \) contains the image of \( 1 \in R \).

Then \( \hat{U} = A_X \), that is, \( U \) is a basis for \( A_X \) over \( S \).

Proof Suppose that \( U \) is not a basis. Then there is a minimal generator \( g \) of \( A_X \) over \( S \) independent from \( U \). We can find a positive integer \( N \) such that \( x^N g \) maps to zero in the Rao module, hence \( x^N g \) belongs to the image
of $R/I_X$ and therefore to $\hat{U}$, since the latter is an $R$-submodule containing $1$. But then there is a relation over $S$ between $U$ and $g$, a contradiction.

**Proof of Theorem 1.7.1**

We may assume that $t \geq 1$ (either because a zero dimensional scheme is ACM, and in this case the theorem is almost trivial, or because we could replace $X$ with the projective cone over $X$). Fix a linear space $L \subseteq \mathbb{P}^N$ of codimension $t + 1$ which does not meet $X$ and let $S = S_L$. First note that $h_X(n) \geq \mu(n)$ for all $n$: indeed, $h_X(n)$ is the number of generators in degree $n$ of $A_X$ over $S$, which are at least as many as the generators in degree $n$ of $A_X$ over $R$. Since $M_X$ is a quotient of $A_X$, we see that $h_X(n) \geq \mu(n)$ for all $n$. The last statement in the Theorem follows now from Corollary 1.4.12 which says that $e(X) + t = 1$ is the largest integer $n$ such that $h_X(n) > 0$. For the same reason, to show that $e(X) + t + 1 \geq 0$ it is enough to prove that $h_X(0) > 0$. We are actually going to prove that $h_X(0) \geq 1 + \mu(0)$: the constant function one is a minimal generator of $A_X$ over $R$ (if it were a combination of elements of negative degrees, it would be nilpotent, which is nonsense) which maps to zero in the Rao module, hence $A_X$ has at least $1 + \mu(0)$ minimal generators over $R$ in degree zero, thus

$$h_X(0) \geq 1 + \mu(0).$$

Next we consider

$$B = A_X \otimes_S k$$

which is a finite length module over $T = R \otimes_S k$; if we write $B_n$ for the $n^{th}$-graded piece of $B$ and we let $\nu(n)$ denote the dimension of the image of the multiplication map $B_{n-1} \otimes T_1 \to B_n$, we have for $n \geq 1$

$$h_X(n) = \dim_k B_n = \nu(n) + \mu(n) \quad \text{(1.5)}$$

Thus to prove the first statement we have to show that $\nu(n) \geq 1$ for $1 \leq n \leq e + t + 1$. Suppose by way of contradiction that $\nu(n) = 0$ for some $n$ with $1 \leq n \leq e + t + 1$. Then the $S$-submodule of $A_X$ generated by the elements of degree less than or equal to $n - 1$ is an $R$-submodule of $A_X$; since $n - 1 \geq 0$, it follows from Lemma 1.7.5 that $A_X$ is generated over $S$ by elements of degree less or equal to $n - 1$, hence $h_X(k) = 0$ for $k \geq n$. This is impossible because $h_X(e + t + 1) > 0$ by Corollary 1.4.12. This proves the first statement.

The proof of the second statement of the Theorem is more complicated and we'll break it into several steps. Assume that $h_X(l) = 1 + \mu(l)$ for some $l$ satisfying $1 \leq l \leq e + t + 1$. Then by equation (1.5) we have

$$\dim_k \text{Image } (B_{l-1} \otimes T_1 \to B_l) = 1 \quad \text{(1.6)}$$
We define $H$ to be the $R$ submodule of $A_X$ generated by the homogeneous elements of degree $\leq l - 1$, and we let

$$D = H \otimes_S k$$

Note that the inclusion $H \hookrightarrow A_X$ induces a map $D \rightarrow B$ which is an isomorphism in degrees $\leq l - 1$ because $H$ and $A_X$ are equal in degrees $\leq l - 1$.

**Step 1**

$$\dim_k D_l = 1 \quad \text{and} \quad D_l \hookrightarrow B_l$$

**Proof** By (1.6) we have $\dim_k D_l \leq 1$. If we had $D_l = 0$, then the $S$-submodule of $A_X$ generated by the homogeneous elements of degree $\leq l - 1$ would be all of $H$ and by Lemma 1.7.5 we would have $H = A_X$. But then $h_X(l) = 0$, contradicting the first statement of the Theorem. Hence $\dim_k D_l = 1$. The map $D_l \rightarrow B_l$ is injective because $H$ and $A_X$ coincide in degrees $\leq l - 1$.

**Step 2** There is an integer $a \geq 0$ such that

$$\dim_k D_n = \begin{cases} 1 & \text{if } l \leq n \leq l + a \\ 0 & \text{if } n \geq l + a + 1 \end{cases}$$

**Proof** Since $\dim_k D_l = 1$, if $z \in R_1$ is the preimage of a general linear form in $T_1$, we have

$$D_l = z D_{l-1}$$

Therefore $z D_l = z T_1 D_{l-1} = T_1 D_l = D_{l+1}$ as $D$ is generated over $T$ by its elements of degree $\leq l - 1$. By induction we see that $z D_{n-1} = D_n$ for all $n \geq l$. This proves our claim. We fix such a linear form $z$ through the rest of the proof.

**Step 3** Assume that $a = 0$, i.e., that $D_{t+1} = 0$. Then $l = e + t + 1$ and $h_X(e + t + 1) = 1$ and $\mu(e + t + 1) = 0$.

**Proof** By Step 1 $D$ is a submodule of $B$ and hence $H$ is a free $S$-submodule of $A_X$. By Lemma 1.7.5 it follows that $H = A$ and $D = B$. Hence $h_X(n) = \dim_k D_n$ for all $n$. Since $a = 0$, we see that $h_X(l) = 1$ and $h_X(l + 1) = 0$. Now the first statement of the Theorem gives the desired conclusion.

From now on we assume $a \geq 1$: by Step 3 this happens if $l \leq e + t$ and it is the difficult case of the Theorem. We define $H^{(n)}$ to be the $S$-submodule
of $H$ generated by elements of degree $\leq n$. Clearly, $H^{(n-1)} \subseteq H^{(n)}$ and $H^{(n)} = H$ for $n \geq l + a$. By Step 2

(1.7) $H^{(n)} = H^{(n-1)} + z H^{(n-1)}$ for $n \geq l$

We define $E$ to be the kernel of the surjective map:

$$z : H^{(l-1)} \to H^{(l)} / H^{(l-1)}$$

By Step 1 $H^{(l)} / H^{(l-1)}$ is the $S$-direct summand of $A_X$ generated by the homogeneous elements of degree $\leq l - 1$. Hence $H^{(l-1)}$ and $E$ are free $S$-modules. Furthermore, by definition of $E$, we have

(1.8) $z E \subseteq H^{(l-1)}$

Next we consider the dual module of $D$:

$$D^* = \text{Hom}_k(D, k)$$

Recall that $D^*$ is a graded $T$-module with grading given by $D^*_n = \text{Hom}_k(D_{-n}, k)$. By Step 2 we have

$$\dim_k D^*_j = 1 \text{ if } 0 \leq j \leq a$$

If $\phi \in D^*_{j-l-a}$ is nonzero, $z^j \phi$ is a $k$-basis of $D^*_j$ for each $j$ satisfying $0 \leq j \leq a$.

**Step 4** Let $U = \{ u \in T_1 : u \phi = 0 \}$. Then $UD_n = 0$ for all $n \geq l - 1$.

**Proof** Pick $u \in U$ and $d \in D_n$: if $n \geq l + a$, then $D_{n+1} = 0$ thus $ud = 0$.

On the other hand, if $l - 1 \leq n \leq l + a - 1$, then $D_{n+1}$ has dimension one and a basis for its dual is $z^j \phi$ where $0 \leq j = l + a - n - 1 \leq a$. We have

$$z^j \phi(ud) = u \phi(z^j d) = 0$$

hence $ud = 0$

**Step 5** There is a subspace $W \subseteq R_1$ of dimension $N - t - 1$ with the following property:

$$w H \subseteq E \text{ for all } w \in W$$

**Proof** Let $U$ be as in Step 4. Then $U$ is the kernel of the multiplication map

$$D^*_j \otimes T_1 \to D^*_j$$
hence $U$ has codimension one in $T_1$, hence dimension $N - t - 1$. Therefore it is enough to show that every $u \in U$ has a unique preimage $w \in R_1$ such that

$$wH \subseteq E$$

Fix $u \in U$ and a preimage $v \in R_1$ of $u$. We first note that

$$vH^{(n)} \subseteq H^{(n)} \quad \text{for } n \geq l - 1$$

since by Step 4 $uD_n = 0$ for $n \geq l - 1$. In particular, we have a commutative diagram

$$
egin{array}{ccc}
0 & \rightarrow & E \\
\downarrow & & \downarrow v \\
0 & \rightarrow & E/H^{(l-1)} \\
\end{array}
$$

from which we see that $vE \subseteq E$. Hence multiplication by $v$ defines a degree one homogeneous endomorphism of $H^{(l-1)}/E$, which is a free $S$-module of rank one being isomorphic to $H^{(l)}/H^{(l-1)}$ by definition of $E$. If $1$ is a generator of $H^{(l-1)}/E$, we have

$$v \cdot 1 = x \cdot 1$$

for some $x \in S_1$. We let $w = v - x$; by construction, $w$ is the unique preimage of $u$ with the property $wH^{(l-1)} \subseteq E$. We now need to show that $wH \subseteq E$, or equivalently that

$$wH^{(n)} \subseteq E \quad \text{for } n \geq l - 1$$

We prove this last statement by induction: assuming $wH^{(l-1+j)} \subseteq E$ for some $j \geq 0$ we show that $wH^{(l+j)} \subseteq E$ as well. Thus assume $wH^{(l-1+j)} \subseteq E$; by definition we have $z^jH^{(l-1)} \subseteq H^{(l-1+j)}$, hence:

$$wz^jH^{(l-1)} \subseteq wH^{(l-1+j)} \subseteq E$$

Now using equality (1.7) we get

$$wz^jH^{(l)} = wz^j(H^{(l-1)} + zH^{(l-1)}) \subseteq w(H^{(l-1+j)} + zH^{(l-1+j)}) \subseteq E + zE \subseteq H^{(l-1)}$$

as $zE \subseteq H^{(l-1)}$ by definition of $E$. We have therefore a commutative diagram:

$$
egin{array}{ccc}
H^{(l-1)}/E & \rightarrow & E/E \\
\downarrow z & & \downarrow z \\
H^{(l)}/H^{(l-1)} & \rightarrow & H^{(l-1)}/E
\end{array}
$$
where the vertical map on the left is an isomorphism. It follows that the horizontal map on the bottom is the zero map, that is
\[ wz^j H^{(l)} \subseteq E. \]

From this we deduce:
\[ wH^{(l+j)} = w(H^{(l-1+j)} + z^j H^{(l)}) \subseteq E. \]

**Step 6** Let \( W \) be as in Step 5. For all \( n \leq t + 1 - l \) and all \( \xi \in H^0(\omega_X(n)) \), we have
\[ w\xi = 0. \]

**Proof** By definition we have \( R/I_X \subseteq H \subseteq A_X \); this by Proposition 1.3.5 implies that
\[ \Omega_X \cong \text{Hom}_S(H, S(-t + 1)). \]
as \( R \)-modules. Pick \( \xi \in (\Omega_X)_n = H^0(\omega_X(n)) \). We can think of \( \xi \) as being an homogeneous homomorp \( H \to S \) of degree \( n - t - 1 \). Now any element in a basis of the free \( S \)-submodule \( E \) of \( H \) has degree \( l - 1 \) by construction, so any nonzero homogeneous morphism \( E \to S \) has degree greater than \( -l \). Therefore, if \( n \leq t + 1 - l \), the degree of \( \xi \) is at most \( -l \) and the restriction of \( \xi \) to \( E \) is zero. Now any \( w \in W \) maps \( H \) into \( E \) by Step 5, hence
\[ w\xi(H) = \xi(wH) \subseteq \xi(E) = \{0\}. \]

**Step 7** We have
\[ h_X(n) = 1 \text{ for } l \leq n \leq e(X) + t + 1 \]
Furthermore there is a linear subspace \( M \subseteq \mathbb{P}^N \) such that \( X \) contains a hypersurface in \( M \) of degree \( e + t + 2 \).

**Proof** We let \( M \) be the \( t + 1 \) dimensional linear subspace of \( \mathbb{P}^N \) defined by \( W \). By Step 6, the equations of \( M \) kill \( (\Omega_X)_{-e} \), hence \( M \cap X \) has dimension \( t \). We consider the beginning of the Koszul resolution of \( O_M \):
\[ O_{\mathbb{P}^N}(-1)^{N-t-1} \to O_{\mathbb{P}^N} \to O_M \to 0 \]
Tensoring with \( O_X \) and applying the functor \( \text{Hom}(\cdot, \omega_L) \) we obtain an exact sequence:
\[ 0 \to \omega_{M \cap X} \to \omega_X \to \omega_X(1)^{N-t-1} \]
where the last map is given by a basis of $W$. Now we use Step 6 to conclude that
\[ H^0(\omega_{M \cap X}(n)) \cong H^0(\omega_X(n)) \text{ for } n \leq t + 1 - l \]
If we let $F$ denote the largest hypersurface of $M$ contained in $M \cap X$, then $\omega_F = \omega_{M \cap X}$ because $F$ and $M \cap X$ coincide in dimension $t$. Thus we have
\[ H^0(\omega_F(n)) \cong H^0(\omega_X(n)) \text{ for } n \leq t + 1 - l \]
By Proposition 1.4.9 this implies
\[ h_X(n) = h_F(n) \text{ for } n \geq l. \]
Now $F$ is a hypersurface in $M$, so its spectrum is $\{0, 1, \ldots, \deg F - 1\}$. We conclude that $\deg F - 1 = e(X) + t + 1$ and
\[ h_X(n) = 1 \text{ for } l \leq n \leq e(X) + t + 1 \]
which finishes the proof of Step 7 and of the Theorem, since $h_X(n) = 1$ implies $\mu(n) = 0$ by the first part of the Theorem.

Next we are going to study those subschemes for which the index of speciality $e$ is as small (resp. as big) as possible.

**Corollary 1.7.6** Let $X \subseteq \mathbb{P}^N$ be a quasi ACM subscheme of dimension $t$ and degree $d$. Then
\[ -t - 1 \leq e(X) \leq d - t - 2. \]
Equality holds on the right if and only if $X$ is a hypersurface in a linear subspace of $\mathbb{P}^N$ of dimension $t + 1$ plane. If equality holds on the left, then the support of $X$ is the disjoint union of at most $h_X(0)$ linear spaces of dimension $t$.

**Proof** By Theorem 1.7.1 we have $e + t + 1 \geq 0$ and $0, 1, \ldots, e + t + 1$ are precisely the nonnegative integers which occur in the spectrum at least once. Since the spectrum consists of $d$ elements, we have $e + t + 2 \leq d$.
If $e = d - t - 2$, then the spectrum of $X$ must be $\{0, 1, \ldots, d - 1\}$. Then Theorem 1.7.1 tells us that $X$ is ACM. If $d = 1$, $X$ itself is a linear subspace, hence a hyperplane in a $t + 1$ linear subspace. If $d \geq 2$, the spectrum gives $\dim(R/I_X)_1 = N - 1 - t$, that is, $X$ is contained in the intersection of $N - 1 - t$ hyperplanes. Such intersection is the linear subspace of dimension $t + 1$ we are looking for. Conversely, if $X$ is a hypersurface of degree $d$ in some linear subspace, we know that $sp_X = \{0, 1, \ldots, d - 1\}$ by Corollary 1.5.3.
Finally, assume that $e(X) = -t - 1$. If $t = 0$, then $X$ is ACM and the conditions on $e = -1$ imply $h_X(n) = 0$ for $n \geq 1$. Hence $sp_X = \{0\}$ and $X$ has degree one, so $X$ is a point with the reduced structure. Now assume $t \geq 1$. If $X$ is reduced, then there cannot be any negative integer in the spectrum, and the spectrum doesn’t contain any positive integer either, since $e = -t - 1$. Hence $sp_X = \{0^d\}$, and each connected component of $X$ must have spectrum $\{0\}$. Thus $X$ is the disjoint union of $d$ linear subspaces of dimension $t$. In general, note that, if $Y$ is any quasi ACM subscheme of $X$ of dimension $t$, we have an injective map $\Omega Y \rightarrow \Omega X$; in particular, $e(X) \geq e(Y)$. Thus, if $e(X) = -t - 1$, we must have $e(X_{\text{red}}) = -t - 1$ and hence the support of $X$ is the disjoint union of $h_X(0)$ linear subspaces. Since $e(X) = e(X_{\text{red}}) = -t - 1$, we have 

$$h_{X_{\text{red}}}(0) \leq h_X(0),$$

and the last part of our statement follows.

**Remark 1.7.7** In the case of curves in $\mathbb{P}^3$ we can be more precise. Recall (see [4]) that a quasi-primitive multiplicity structure on a line in $\mathbb{P}^3$ is a curve $C \subseteq \mathbb{P}^3$ whose support is a line and which is almost everywhere locally contained in a smooth surface, that is, the embedded dimension of $C$ is (at most) 2 at all but finitely many points. We claim that a curve $C$ in $\mathbb{P}^3$ has $e(C) = -2$ if and only if each connected component of $C$ is a quasi-primitive multiplicity structure on a line which does not contain a planar double structure. Suppose first that $e(C) = -2$. Since $e(C)$ equals the maximum value for $e(D)$ as $D$ varies among the connected components of $C$, we see from the Corollary that each connected component $D$ of $C$ is a multiplicity structure on a line $L$ with $e(D) = -2$. The second infinitesimal neighborhood $L_2$ of $L$ has the same cohomology as the twisted cubic curve, in particular it has $e = -1$, hence $D$ cannot contain $L_2$, that is, $D$ is quasi-primitive ([4]). Similarly, since a conic has $e = -1$, $D$ cannot contain a planar double structure.

To prove the converse, assume $D$ is a quasi-primitive multiplicity structure on a line $L$ which does not contain a planar double structure. Recall from [4] that there is a chain of inclusions $L = D_1 \subseteq D_2 \subseteq \cdots \subseteq D_d = D$, where $D_k$ is a multiplicity structure of degree $k$, and we have exact sequences:

$$0 \rightarrow \mathcal{L}_k \rightarrow \mathcal{O}_{D_{k+1}} \rightarrow \mathcal{O}_{D_k} \rightarrow 0$$

where $\mathcal{L}_k \in \text{Pic}(L)$ and $\deg \mathcal{L}_k \geq k \deg \mathcal{L}_1$. Now, since $D$ does not contain a planar double structure, we must have $g(D_2) < 0$, that is, $\deg \mathcal{L}_1 \geq 0$. Hence $\deg \mathcal{L}_k \geq 0$ for $1 \leq k \leq d - 1$, and this implies $H^1(\mathcal{L}_k(-1)) = 0$. By induction on $k$ it follows that $e(D_k) = -2$. 

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1.8 Bounds for the Arithmetic Genus of a Space Curve

V. Beorchia [7, 6] has found a bound $P(d, t)$ for the arithmetic genus of a curve $C \subseteq \mathbb{P}^3$ of degree $d$ not contained in any surface of degree less than $t$ when the ground field $k$ had characteristic zero; she has also shown sharpness when $1 \leq t \leq 4$. Her proof depends on a result of Strano which holds only in characteristic zero. In this section we give a new proof of Beorchia’s bound which is characteristic free.

Recall that by definition a curve is a quasi ACM subscheme of $\mathbb{P}^N$ of dimension one. By Proposition 1.4.8 the arithmetic genus $g(C)$ is given in terms of its spectrum by the formula

$$g(C) = \sum_{n \in \mathbb{Z}} (n - 1)h_C(n) + 1$$

We let $s(C)$ denote the smallest degree of a surface containing $C$. Our next task is to find an upper bound for $s(C)$ in terms of the index of speciality $e(C)$ and of the degree $\deg(C)$. This is the first example we encounter of a restriction imposed by the spectrum on the postulation.

Proposition 1.8.1 For a curve $C \subseteq \mathbb{P}^3$ we have:

$$s(C) \leq \deg(C) - e(C) - 2.$$  

Proof We first prove that for a general three dimensional space of linear forms $< x, y, z >$ the number of minimal generators of $A_C$ as a module over $k[x, y, z]$ is bounded by $d - e - 2$ where $d = \deg C$.

We write $R = k[x, y, z, w]$ where $x$ and $y$ are chosen so that the line $L$ of equations $x = y = 0$ does not meet $C$, and we let $S = S_L = k[x, y]$. We know that $F = A_C \otimes S$ is a length $d$ module over $k[z, w]$; if we write $F_n$ for the $n^{th}$-graded piece of $F$, multiplication by a linear form in $k[z, w]$ gives a map $F_n \to F_{n+1}$.

Suppose this map is zero for all linear forms and some $n$ with $0 \leq n \leq e + 1$. Then the $S$-submodule of $A_C$ generated by the elements of degree less than or equal to $n$ is an $R$-submodule of $A_C$; since $n \geq 0$, it follows from Lemma 1.7.5 that $A_C$ is generated over $S$ by elements of degree less or equal to $n$, but this contradicts the fact that $n \leq e + 1$. Thus multiplication by a general linear form, say $z$, in $k[z, w]$ gives a nonzero map $F_n \to F_{n+1}$ for $0 \leq n \leq e + 1$. It follows that

$$\dim_k F \otimes_{k[z]} k \leq \dim_k F - e - 2 = d - e - 2$$
This says that $A_C$ has at most $d - e - 2$ generators over $k[x, y, z]$. Now let $g_1, \ldots, g_m$ where $m \leq d - e - 2$ be a set of minimal generators for $A_C$ over $k[x, y, z]$; the determinant trick applied to multiplication by $w$ on $A_C$ gives us the equation of a surface of degree $m$ containing $C$ and we are done: we can write

$$wg_i = \sum f_{ij}g_j$$

where the $f_{ij}$ are homogeneous polynomials in $k[x, y, z]$. Then the determinant of the matrix $[\delta_{ij}w - f_{ij}]$ annihilates all the $g_i$'s, hence all of $A_C$. Expanding the determinant and taking the homogeneous component of degree $m$, we find a polynomial

$$w^m + h_1(x, y, z)w^{m-1} + \cdots + h_m(x, y, z)$$

which kills $1 \in A_C$ and therefore is the equation of a surface containing $C$.

Remark 1.8.2 The bound in the Proposition is sharp: for all positive integers $s$ and $d$ with $s \leq d$ there are curves $C$ satisfying $s(C) = s$, $\deg(C) = d$ and $e(C) = d - s - 2$. When $s = d$, we can take $L$ to be a line on a smooth surface $F$ of degree greater or equal to $d$ and $C$ in the divisor class of $dL$ on $F$: we have $I_C = (dL, F)$, so $s(C) = d$, while $e(C) = -2$ by Corollary 1.7.6.

If $s < d$, start with $D$, a multiplicity $s$-structure on a line with $s(C) = s$ as above, and note that $D$ is contained in a surface of degree $s$ having a plane $H$ as a component. Let $C$ be the (scheme theoretic) union of $D$ and a general curve $B$ of degree $d - s$ in $H$; then $C$ has degree $d$ and, by construction, $s(D) = s$. On the other hand, $C$ contains a plane curve of degree $d - s + 1$, namely $B$ union the line $L$ supporting $D$; therefore $e(C) \geq e(B \cup L) = d - s - 2$. Combining with the Proposition, we see that $e(C) = d - s - 2$.

We will show later that in characteristic zero every curve $C \subseteq \mathbb{P}^3$ satisfying $e(C) \geq 1$ and $s(C) = \deg(C) - e(C) - 2$ contains a plane curve of degree $\deg(C) - s(C) + 1$.

We can now prove Beorchia’s bound $P(d, t)$ for the arithmetic genus of a curve $C \subseteq \mathbb{P}^3$ of degree $d$ satisfying $s(C) \geq t$. Our proof has the advantage of being characteristic free (Beorchia’s proved her Theorem only in characteristic zero), and doesn’t involve complex calculations. Note that the condition $s(C) \geq t$ implies $d \geq t$ by Proposition 1.8.1. Beorchia [7] has shown that this bound is sharp for $1 \leq t \leq 4$, but we don’t know if this the case for $t \geq 5$. But for higher values of $t$ the problem of finding a sharp bound is open. For $t = 1$ and $t = 2$, the bound has been proven independently in [43], [46] and [19]; in particular, [43] and [46] use the
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theory of the spectrum for sheaves to find the bound $P(d,2)$, and our proof is a generalization of their argument based on Proposition 1.8.1.

Theorem 1.8.3 (Beorchia) Let $C$ be a curve in $\mathbb{P}^3$ of degree $d$ and genus $g$. Assume that $C$ is not contained in any surface of degree less than $t$. Then

$$g \leq P(d,t) = \begin{cases} (t-1)d + 1 - \binom{t+2}{3} & \text{if } t \leq d \leq 2t \\ \frac{1}{2}(d-t)(d-t-1) - \binom{t-1}{3} & \text{if } d \geq 2t+1 \end{cases}$$

Proof Assume that $C$ is not contained in any surface of degree $t-1$. Then

$$h^0(\mathcal{O}_{\mathbb{P}^3}(t-1)) \leq h^0(\mathcal{O}_C(t-1)) = h^1(\mathcal{O}_C(t-1)) + (t-1)d + 1 - g$$

so that a bound for $h^1(\mathcal{O}_C(t-1))$ will give us a bound for the genus. If we have $t \leq d \leq 2t$, Proposition 1.8.1 implies that $e(C) \leq t-2$; hence $h^1(\mathcal{O}_C(t-1)) = 0$ and we obtain

$$g \leq (t-1)d + 1 - \binom{t+2}{3}.$$

Now assume $d \geq 2t+1$; we investigate how large $h^1(\mathcal{O}_C(t-1))$ can be by looking at the spectrum of $C$. We have by Proposition 1.4.8

$$h^1(\mathcal{O}_C(t-1)) = \sum_{n \geq t+1} (n-t)h_C(n).$$

Now, if for an integer $n \geq t+1$ we have $h_C(n) \geq 2$, then by Theorem 1.7.1 the spectrum contains

$$\{0, 1^2, 2^2, \ldots, (t+1)^2\}$$

hence there are at most $d-1-2t$ elements of the spectrum which are greater or equal to $t+1$. Since the spectrum is connected in positive degrees, a spectrum which gives the largest value for $h^1(\mathcal{O}_C(t-1))$ in this case ends with $(t+1)^2, t+2, \ldots, d-t-2$.

On the other hand, if $h_C(t+1) = 1$, by Theorem 1.7.1 we have $h_C(n) = 1$ for $n = t+1, t+2, \ldots, e+2$ and $h_C(n) = 0$ for $n \geq e+3$. also $e+2 \leq d-t$ by Proposition 1.8.1. Therefore in both cases we have

$$h^1(\mathcal{O}_C(t-1)) = \sum_{n \geq t+1} (n-t)h_C(n) \leq \sum_{j=t+1}^{d-t} (j-t) = \frac{1}{2}(d-2t+1)(d-2t)$$
which gives the desired bound for the genus:

\[
g \leq \frac{1}{2}(d - 2t + 1)(d - 2t) + (t - 1)d + 1 - \binom{t+2}{3}
\]

\[
= \frac{1}{2}(d - t)(d - t - 1) + \frac{3}{2}t(t - 1) + 1 - \binom{t+2}{3}
\]

\[
= \frac{1}{2}(d - t)(d - t - 1) - \binom{t-1}{3}.
\]
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Chapter 2

The Spectrum of a Curve in \( \mathbb{P}^3 \)

2.1 The Principal Spectrum

In this section we introduce the principal spectrum of a quasi ACM subscheme \( X \) of \( \mathbb{P}^N \). It should be thought as a well behaved part of the spectrum: the spectrum is the set of integers which occur as degrees of the elements in a basis of \( A_X \) over the ring \( S \); as such it contains the set of integers which occur as degrees of the elements in a basis of the Rao module \( M_X \). The principal spectrum is the difference of these two sets and behaves like the spectrum of an ACM subscheme.

Let \( X \subseteq \mathbb{P}^N \) be a quasi ACM subscheme of dimension \( t \) and let \( L \subseteq \mathbb{P}^N \) be a linear subspace of codimension \( t+1 \) which does not meet \( X \). As usual, we denote by \( R \) the homogeneous coordinate ring of \( \mathbb{P}^N \) and by \( S = S_L \) the polynomial subring of \( R \) generated by the linear equations of \( L \). If \( M_X \) is the Rao module of \( X \), we have an exact sequence

\[
0 \to R_X \to A_X \to M_X \to 0
\]

Tensoring over \( S \) with \( k \cong S/m_S \) we obtain

\[
R_X \otimes k \to A_X \otimes k \to M_X \otimes k \to 0.
\]

We define \( P = P(X, L) \) to be the image of \( R_X \otimes k \) in \( A_X \otimes k \)

**Definition 2.1.1** The principal spectrum \( sp_{X, L} \) of \( X \) relative to \( L \) is the set of integers with multiplicities \( \{ n^{h_{X, L}(n)} \} \) where

\[
h_{X, L}(n) = \dim_k P(X, L)_n
\]
Remark 2.1.2 We have
\[ h_{X,L}(n) = h_X(n) - \mu_{X,L}(n) \] (2.1)
where \( \mu_{X,L} = \dim_k (M_X \otimes k)_n \) is the number of generators in degree \( n \) of \( M_X \) as an \( S \)-module: indeed, by definition of the spectrum we have \( h_X(n) = \dim_k (A_X \otimes k)_n \). In particular, for any \( L \) we have \( h_{X,L}(n) \leq h_X(n) \) for all \( n \), that is, the principal spectrum is contained in the spectrum. Note also that the principal spectrum is never empty: zero always occurs in the principal spectrum because the constant function one is a minimal generator of \( A_X \) which maps to zero in the Rao module.

Remark 2.1.3 It follows from (2.1) that \( X \) is ACM if and only if \( sp_{X,L} = sp_X \) for some (resp. all) \( L \).

Remark 2.1.4 Note that \( T = R \otimes_S k \) is a polynomial ring and that \( P = P(X, L) \) is a quotient of \( T \) by a proper homogeneous ideal. In particular, Macaulay’s Theorem 1.6.4 applies to \( P \). Therefore \( h_{X,L}(0) = 1 \) and
\[ h_{X,L}(n + 1) \leq h_{X,L}(n)^{(n)} \text{ for all } n \geq 1. \]

If \( X \) has codimension two, then \( T \) is a polynomial ring in two variables and Proposition 1.6.11 applies with \( V = P(X, L) \) and by definition we have \( sp_P = sp_{X,L} \). We will write \( s(X, L) \) and \( t(X, L) \) for the invariants \( s(P) \) and \( t(P) \) of \( P \).

Suppose \( X \subseteq Y \subseteq \mathbb{P}^N \) are quasi ACM subschemes of the same dimension. In general, it is not true that the spectrum of \( X \) is a subset of the spectrum of \( Y \): take for example \( X \) to be the union of two disjoint lines in \( \mathbb{P}^3 \), and let \( Y \) be the union of \( X \) with a line meeting each of the lines in \( Y \); we have \( sp_X = \{0^2\} \) while \( sp_Y = \{0, 1^2\} \) since \( Y \) is a deformation with constant cohomology of the twisted cubic curve. However, it is true that the principal spectrum of \( X \) is contained in that of \( Y \).

Proposition 2.1.5 Let \( X \subseteq Y \subseteq \mathbb{P}^N \) be quasi ACM subschemes and suppose \( X \) and \( Y \) have dimension \( t \). Let \( L \) be a linear subspace of codimension \( t + 1 \) which doesn’t meet \( Y \). Then \( sp_{X,L} \subseteq sp_{Y,L} \), that is,
\[ h_{X,L}(n) \leq h_{Y,L}(n) \text{ for all integers } n. \]

Proof The inclusion \( X \subseteq Y \) induces a surjective map \( R_Y \rightarrow R_X \) of the coordinate rings of \( X \) and \( Y \). This map remains surjective upon tensoring over \( S = S_L \) with \( k = S/m_S \) and hence it induces a surjective map
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$P(Y, L) \to P(X, L)$. Hence $\dim P(Y, L)_n \geq \dim P(X, L)_n$ for all $n$, which is what we needed to show.

Our next task is to show that there is a dense open subset $U$ of the Grassmannian variety of codimension $t + 1$ linear subspaces of $\mathbb{P}^N$ such that the principal spectrum of $X$ relative to any $L \in U$ is as big as possible and independent of the choice of $L$. The argument depends on the following Lemma:

**Lemma 2.1.6** Let $M$, $N$ and $W$ be finite dimensional $k$-vector spaces and let $\psi : W \otimes_k M \to N$ be a linear map. For each $V$ in the Grassmannian variety $G(l, W)$ of $l$-dimensional subspaces of $W$ define $\psi_V : V \otimes_k M \to N$ by restricting $\psi$ to $V \otimes_k M$. Then the function $V \mapsto \text{rank } \psi_V$ is lower semicontinuous on $G(l, W)$.

**Proof** The problem is local on $G = G(l, W)$ so we can restrict our attention to the standard open affine subschemes of $G$. Fix $V \in G$ and choose a complement $Z$ of $V$ in $W$. Define a linear map

$$
\phi : \text{Hom}_k (V, Z) \to \text{Hom}_k (V, W)
$$

by mapping $A$ to $1_V + A$. The morphism $A \mapsto V_A = \phi(A)(V)$ is an isomorphism of the affine space $\text{Hom}_k (V, Z)$ onto an open neighborhood of $V = V_0$ in $G$. Now consider the linear map

$$
\chi : \text{Hom}_k (V, Z) \to \text{Hom}_k (V \otimes M, N)
$$

which assigns to $A \in \text{Hom}_k (V, Z)$ the composite

$$
\chi(A) : V \otimes M \xrightarrow{\phi(A) \otimes 1_M} W \otimes M \xrightarrow{\psi} N.
$$

Since $V_A$ is the image of $\phi(A)$ in $W$, the linear maps $\chi(A)$ and $\psi_{V_A}$ have the same image in $N$, therefore

$$
\text{rank } \chi(A) = \text{rank } \psi_{V_A} \quad \text{for all } A \in \text{Hom}_k (V, Z)
$$

Now the problem is reduced to the following: given finite dimensional vector spaces $H$, $M$ and $N$ and a linear map

$$
\chi : H \to \text{Hom}_k (M, N)
$$

the function $H \to \mathbb{N}$ which maps $A \in H$ to $\text{rank } \chi(A)$ is upper semicontinuous. But this is easy: we may think of $\chi$ as a matrix of linear forms on
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$H$, and the minors of $\chi$ as polynomials in the symmetric algebra over $H^*$. Then the set of $A \in H$ such that

$$\text{rank } \chi_A < r$$

is precisely the zero set of the $r \times r$ minors of $\chi$, hence it is closed in the Zariski topology. This says that $\chi$ is lower semicontinuous and concludes the proof.

**Proposition 2.1.7** Let $X \subseteq \mathbb{P}^N$ be a quasi ACM subscheme of dimension $t$. Let $Gr$ be the Grassmannian variety of codimension $t+1$ linear subspaces of $\mathbb{P}^N$. There is an open dense subset $U \subseteq Gr$ such that for all $L \in U$, $L$ does not meet $X$ and

$$\text{sp}_{X,L} \subseteq \text{sp}_{X,L'}$$

for any $L' \in Gr$ which does not meet $X$. In particular, $\text{sp}_{X,L} = \text{sp}_{X,L'}$ for all $L, L' \in U$.

**Proof** Since $X$ has dimension $T$, the set $U_0$ of linear spaces $L$ in $Gr$ which do not meet $X$ is open and dense in $Gr$. For all $L \in U_0$ the principal spectrum $\text{sp}_{X,L}$ is defined as a subset of $\text{sp}_{X}$. In particular, there are only finitely many integers $n$ for which $h_{X,L}(n) > 0$ for some $L \in U_0$. So it is enough to show that for any fixed $n$ the function $L \mapsto h_{X,L}(n)$ is lower semicontinuous on $U_0$. Now we have by Remark 2.1.2

$$h_{X,L}(n) = h_X(n) - \mu_{X,L}(n)$$

where $h_X(n)$ is independent of $L$ and $\mu_{X,L} = \dim_k (M_X \otimes_k S)_n$ is the number of generators in degree $n$ of $M_X$ as an $S = S_L$-module. If $V_L$ is the space of linear forms in $R_1$ which vanish on $L$, then

$$\mu_{X,L}(n) = \dim_k M_n - \dim_k \text{Im} (V_L \otimes_k M_{n-1} \rightarrow M_n)$$

where $M = M_X$ is the Rao module and $V_L \otimes_k M_{n-1} \rightarrow M_n$ is the restriction to $V_L \otimes_k M_{n-1}$ of the multiplication map $R_1 \otimes_k M_{n-1} \rightarrow M_n$. Since the morphism $L \rightarrow V_L$ gives an isomorphism of $Gr$ with the Grassmannian $Gr^*$ of $t+1$ dimensional linear subspaces of $R_1$, it is enough to show that the function

$$V \mapsto \dim_k \text{Image} (V_L \otimes_k M_{n-1} \rightarrow M_n)$$

is lower semicontinuous on $Gr^*$. But this follows from Lemma 2.1.6 applied to the linear spaces $M_{n-1}, M_n$ and $R_1$ and the multiplication map $R_1 \otimes_k M_{n-1} \rightarrow M_n$. 
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Definition 2.1.8 Let $X$ be a quasi ACM subscheme of $\mathbb{P}^N$ and let $U$ be as in Proposition 2.1.7. The principal spectrum of $X$ is the principal spectrum of $X$ relative to any $L \in U$.

We are now going to consider the case of curves. Recall that we have defined a curve to be a locally Cohen-Macaulay equidimensional projective scheme of dimension one. By Proposition 1.2.8 a subscheme $C \subseteq \mathbb{P}^N$ is a curve if and only if it is quasi ACM of dimension one. Now let $C \subseteq \mathbb{P}^N$ be a curve and let $L$ be a codimension two linear subspace that doesn’t meet $C$. If $H$ is a hyperplane containing $L$, then $H$ contains no component of $C$ because $L \cap C = \emptyset$. The scheme theoretic intersection $C.H$ of $C$ and $H$ has therefore dimension zero, hence is ACM and has a well defined spectrum. We are now going to investigate the relations between $sp_C$, $sp_{C.H}$ and $sp_{C,L}$. We will fix coordinates so that $H$ has equation $x = 0$ and $L$ has equations $x = y = 0$. We introduce the following notation: we let $M = M_C$ be the Rao module of $C$ and we define

$$Q = Q_x = \text{coker}(x : M(-1) \to M)$$

$$K = K_{xy} = \ker(y : Q(-1) \to Q)$$

We denote by $q(n)$ and $k(n)$ the dimensions of $Q_n$ and $K_n$ respectively. Note that $K_n$ is a subspace of $Q_{n-1}$, hence $k(n) \leq q(n-1)$.

Proposition 2.1.9 Let $C \subseteq \mathbb{P}^N$ be a curve. Let $L = (x, y)_0$ be a codimension two linear subspace which does not meet $C$ and let $H = (x)_0$ be a hyperplane containing $L$. Then

$$h_C(n) = h_{C,H}(n) + \partial q(n)$$

$$h_{C,H}(n) = h_{C,L}(n) + k(n) \geq h_{C,L}(n)$$

Proof We write $A$ for $A_X$ and $Z$ for $C.H$. We have already shown in Proposition 1.4.17 that there is an exact sequence

$$(2.2) \quad 0 \to R_Z \to A/xA \to Q \to 0$$

and that this implies

$$h_C(n) = h_Z(n) + \partial q(n).$$

To study the relation between $sp_Z$ and $sp_{C,L}$ we need to introduce some more notation. We define

$$K = K_{xy} = \ker(y : Q(-1) \to Q)$$
CHAPTER 2. THE SPECTRUM OF A CURVE IN $\mathbb{P}^3$

\[ O = O_{xy} = \text{coker}(y : Q(-1) \to Q) \]

The exact sequence (2.2) gives us a commutative diagram with exact rows:

\[
\begin{array}{cccccc}
0 & \to & R_Z(-1) & \to & A/xA(-1) & \to & Q(-1) & \to & 0 \\
\downarrow y & & \downarrow y & & \downarrow y & & \\
0 & \to & R_Z & \to & A/xA & \to & Q & \to & 0
\end{array}
\]

The first two vertical arrows are injective because $x, y$ form an $A$-regular sequence. The Snake’s Lemma gives us an exact sequence which we break into the following two short exact sequences:

\[(2.3) \quad 0 \to K \to R_Z/yR_Z(-1) \to P \to 0\]

\[(2.4) \quad 0 \to P \to A/(x, y)A \to O \to 0\]

The latter sequence shows us that $P = P(X, L)$ is the module which defines $\text{sp}_{C, L}$, so that $\dim P_n = h_{C, L}(n)$; the former sequence gives us the desired formula:

\[ h_Z(n) = \dim(R_Z)_n - \dim(R_Z)_{n-1} = h_{C, L}(n) + \dim K_n. \]

**Corollary 2.1.10**

- $C$ is arithmetically Cohen Macaulay if and only if $\text{sp}_C = \text{sp}_{C, H} = \text{sp}_{C, L}$.
- If $H^1(J_C(l)) = 0$ for $l \leq s(C, L) - 1$, then $s(C, L) = s(C)$. In particular, a connected reduced curve with $s(C, L) = 1$ is contained in a hyperplane.

**Proof** The first statement follows immediately from Proposition 2.1.9. As for the second one, in the notation of the proof of 2.1.9, we are assuming $M_l = 0$ for $l \leq s(C, L) - 1$; hence $K_l = 0$ for $l \leq s(C, L)$. The above formula now shows $h_{C, H}(l) = h_{C, L}(l)$ for $l \leq s(C, L)$, thus $s(C, H) = s(C, L)$. Now let $m = s(C, H)$. We have $M_l = 0$ for $l \leq m - 1$, hence an exact sequence

\[ H^0(\mathbb{P}^N, J_C(m)) \to H^0(\mathbb{P}^N, J_{C, H}(m)) \to M_{m-1} = 0 \]

from which we see that $s(C) \leq m$. But we always have $s(C) \geq s(C, H)$, hence equality must hold.

**Example 2.1.11** In general, we have $s(C, L) \leq s(C, H) \leq s(C)$. Note also that

\[ s(C, L) \leq t(C, L) \leq e(C) + 3 \]
as the spectrum relative to $L$ is contained in $sp_C$, the largest element of which is $e(C) + 2$. Equality can hold everywhere: for example, pick $C$ to be a line. On the other hand, a curve with $s(C) > e(C) + 3$ necessarily has $s(C) > s(C, L)$.

**Example 2.1.12** We give now some examples of reduced, connected curves in $P^3$ with $s(C, L) = 2$ for $L$ a general line. A trivial example is given by reduced, connected curves with $s(C) = 2$. We now describe a few more. Let $C$ be a connected reduced curve of degree $d$ and genus zero. Then $sp_C = \{0, 1^{d-1}\}$. For $d = 1, 2$, $C$ is ACM so $sp_C = sp_{C,L}$. On the other hand, if $d \geq 3$, $C$ cannot be contained in a plane, and then Corollary 2.1.10 tells us that $s(C, L) \geq 2$. Since $e(C) = -1$, this implies $sp_{C,L} = \{0, 1^2\}$ for all lines $L$ which do not meet $C$. If we pick $C$ to be a rational smooth connected curve of degree 5, the general plane section will have spectrum $\{0, 1^2, 2^2\}$ by the uniform position theorem. Note that $C$ need not be contained in a quadric surface.

For reduced connected curves of genus one and degree greater or equal than four, the same argument shows $s(C, L) = 2$. In characteristic zero we will later prove that $sp_{C,L} = \{0, 1^2, 2\}$ for general $L$: we will show that the largest integer appearing in $sp_{C,L}$ for $L$ general is $e(C) + 2$.

A nonspecial curve of degree $d$ and genus 2 has $sp_C = \{0, 1^{d-3}, 2^2\}$, thus $s(C, L) = 2$ for all $L$. We conjecture that the principal spectrum in this case is $\{0, 1^2, 2^2\}$ and that our list of integral curves with $s(C, L) = 2$ is complete.

### 2.2 Variation of the Spectrum under Biliaison

In this section we study the variation of the spectrum under an elementary biliaison. We refer the reader to [26] for a treatment of liaison form a geometric point of view.

Given a set of integers with multiplicities $sp = \{n^{h_{sp}(n)}\}$ and an integer $l$, we define $sp(-l)$ to be the set of integers with multiplicities $\{n^{h_{sp}(n-l)}\}$.

**Proposition 2.2.1** Let $C$ and $D$ be curves on an ACM surface $F$ in $\mathbb{P}^N$, and suppose that there is a positive integer $l$ such that

$$ J_{D,F} \cong J_{C,F}(-l) $$

Then

$$ sp_D = sp_C(-l) \cup sp_{F,l} $$
where $F_l$ denotes the scheme theoretic intersection of $F$ with a hypersurface of degree $l$ which meets $F$ properly. Furthermore, if $H$ is a hyperplane meeting $C$ and $D$ properly (resp. if $L$ is a line which meets neither $C$ nor $D$), the formula holds with $sp_{D,H}$ and $sp_{C,H}$ (resp. $sp_{D,L}$ and $sp_{C,L}$) in place of $sp_D$ and $sp_C$.

**Proof** The statement makes sense because, as $F$ is ACM, $F_l$ is also ACM and the spectrum of $F_l$ is well defined. The fact that $F$ is ACM implies $H^1(\mathbb{P}^N, J_{F_l}) = 0$ and

$$H^1(\mathbb{P}^N, O_F) \cong H^2(\mathbb{P}^N, J_F) = 0.$$ 

This together with our hypothesis $J_{D,F} \cong J_{C,F}(-l)$ gives an isomorphism

$$M_D \cong M_C(-l)$$

for the Rao modules of $C$ and $D$. Now by Proposition 2.1.9 it is enough to prove our statement for the spectrum. We compute:

$$h^0 O_D(n) = h^1 J_{D,F}(n) - h^0 J_{D,F}(n) + h^0 O_F(n)$$

$$= h^1 J_{C,F}(n-l) - h^0 J_{C,F}(n-l) + h^0 O_F(n)$$

$$= h^0 O_C(n-l) - h^0 O_F(n-l) + h^0 O_F(n)$$

But $O_F(-l) \cong J_{F_l,F}$, hence

$$h^0 O_D(n) = h^0 O_C(n-l) + h^0 O_{F_l}(n)$$

Now recall that for a curve $X$ we have $h_X(n) = \partial^2 h^0 O_X(n)$, so taking second differences we obtain:

$$h_D(n) = h_{C}(n-l) + h_{F_l}(n)$$

which is what we needed to prove.

**Remark 2.2.2** We borrow the following definition of an elementary biliaison from [26] page 318; an equivalent definition, more elementary but less geometrically clear, is given in [32].

Let $C$ and $D$ be curves in $\mathbb{P}^N$. We say that $D$ is obtained from $C$ by an elementary biliaison (or double linkage) of height $h$, for some $h \in \mathbb{Z}$, if there is a complete intersection surface $F$ containing $C$ and $D$, such that $D$ is linearly equivalent to $C + hH$ as generalized divisors on $F$, where $H$ is a hyperplane section of $F$. Furthermore, if $h$ is positive (resp. negative), we say that $C$ bilinks up (resp. down) on $F$. Note that saying that $D$ is
2.2. VARIATION OF THE SPECTRUM UNDER BILIAISON

linearly equivalent to $C + hH$ on $F$ means that there is an isomorphism $\mathcal{J}_{D,F} \cong \mathcal{J}_{C,F}(-h)$.

Since a complete intersection is ACM, it follows from the above Proposition that, if $D$ is obtained from $C$ by an elementary biliaison of height $h \geq 1$ on the surface $F$, then

$$sp_D = sp_C(-h) \cup sp_{F,h}.$$  

**Example** As a first application, we compute the spectrum of a curve $C$ on a nonsingular quadric surface $Q$ in $\mathbb{P}^3$. We denote by $F_{ab}$ the complete intersection of two surfaces of degree $a$ and $b$ in $\mathbb{P}^3$. If $a \leq b$, the spectrum of $F_{ab}$ is given by:

$$sp_{F_{ab}} = \{0, 1^2, \ldots, (a - 1)^a, (b - 1)^{a-1}, \ldots, (a + b - 3)^2, a + b - 2\}.$$  

If $C$ is a divisor of type $(a, 0)$ on $Q$, then $C$ is a deformation with constant cohomology of a disjoint union of $a$ lines, hence $sp_C = \{0^a\}$. If $C$ has type $(a, b)$ with $a \geq b > 0$, $C$ is obtained by an elementary biliaison of height $b$ on $Q$ from a curve $V$ of type $(a - b, 0)$. Hence

$$sp_C = sp_V(-b) \cup sp_{F_{2b}} = \{b^{a-b}\} \cup \{0, 1^2, \ldots, (b - 1)^2, b\} = \{0, 1^2, \ldots, (b - 1)^2, b^{a-b+1}\}.$$  

Similarly, the principal spectrum of a curve of type $(a, 0)$ with $a > 0$ is $\{0\}$, so the principal spectrum of a curve $C$ of type $(a, b)$ with $a > b > 0$ is

$$sp_{C,L} = \{0, 1^2, \ldots, (b - 1)^2, b^2\}$$

Finally, if $C$ is the disjoint union of $a$-lines on $Q$ with $a \geq 3$, the general plane $H$ meets $C$ in $a$ distinct points no three of which are collinear; it follows as above that, if $C$ has type $(a, b)$,

$$sp_C.H = \begin{cases} 
\{0, 1^2, \ldots, (k - 1)^2, k\} & \text{if } a + b = 2k \text{ is even} \\
\{0, 1^2, \ldots, (h - 1)^2, h^2\} & \text{if } a + b - 1 = 2h \text{ is even}
\end{cases}$$

Note that, for all $C \subseteq Q$, if $L$ and $H$ are respectively a general line and a general plane, we have

$$sp_{C,L} = sp_C \cap sp_C.H$$

**Sketch of an Example** One can make a similar computation on all quadric surfaces in $\mathbb{P}^3$. Let’s consider the union $F$ of two distinct planes.
CHAPTER 2. THE SPECTRUM OF A CURVE IN \( \mathbb{P}^3 \)

Minimal curves on \( F \) have been found by Hartshorne: see [26] Example 5.7. Every non ACM curve on \( F \) is obtained from a minimal curve by an elementary biliaison on \( F \). The minimal curves \( C \) have spectrum \( \text{sp}_C = \{ \alpha, 0, 1, \ldots, \deg(C) - 2 \} \) for some integer \( \alpha \leq 0 \), their principal spectrum relative to a general line \( L \) is \( \text{sp}_C, L = \{ 0, 1, \ldots, \deg(C) - 2 \} \) and, for a general plane \( H \) and \( \deg(C) \geq 3 \), \( \text{sp}_{C,H} = \{ 0, 1^2, 2, \ldots, \deg(C) - 2 \} \). Furthermore, any curve with spectrum \( \text{sp}_C = \{ \alpha, 0, 1, \ldots, \deg(C) - 2 \}, \alpha \leq 0 \), is a minimal curve contained in the union of two distinct planes, and any such spectrum arises as the spectrum of a curve. This also classifies curves \( C \) with \( e(C) = \deg(C) - 4 \): any such \( C \) is either an ACM curve obtained from a plane curve of degree \( d - 2 \) by an elementary biliaison of height \( 1 \) on a quadric surface or a minimal curve curve on the union of two distinct planes. Note that by Theorem 1.7.1 a curve with \( e(C) = \deg(C) - 4 \) either contains a plane curve of degree \( d - 1 \) or is an integral ACM curve of degree 3 and (arithmetic) genus 0 or is the complete intersection of two quadric surfaces.

Hartshorne (unpublished work) has shown how to compute the postulation of a curve \( C \) contained in a double plane \( 2H \); his method can be extended to compute the whole cohomology of \( J_C \) and to characterize minimal curves. These have a spectrum of the following form:

\[
\text{sp}_C = \{ (-t)^{a_1}, (1-t)^{a_2-1}, \ldots, (-1)^{a_t}, 0^{1+a_0}, 1, \ldots, p-1 \}
\]

where \( p \geq \deg(C)/2 \) is the degree of the intersection of \( C \) with the reduced plane \( H; \ t \geq 0, \ a_i \geq 0, \ a_i \geq 1, \sum a_i = s \leq p \). When \( s = 1 \), \( C \) is contained in the union of two distinct planes and the spectrum of \( C \) is \( \{ (-t), 0, 1, \ldots, p-1 \} \) where \( p = \deg(C) - 1 \), as we already know.

The following Proposition gives a criterion for bilinking down a curve \( C \) on a surface \( Q \) in \( \mathbb{P}^3 \); we argue as in [32] pp. 64-65, but we go a little further.

**Proposition 2.2.3 (Biliaison Trick)** Let \( C \) be a curve in \( \mathbb{P}^3 \). Let \( Q \) be a surface of degree \( s \) containing \( C \). Suppose there exists a nonzero element \( \xi \in H^0 \omega_C(-m) \) such that \( f \cdot \xi \neq 0 \) for every form \( f \) of degree \( l < s \) which divides the equation of \( Q \). Then either there is a curve \( C_0 \subseteq Q \) such that \( C \) is obtained from \( C_0 \) by an elementary biliaison on \( Q \) of height \( m + 4 - s \), or \( m + 4 - s \geq 1 \) and \( C \) is the complete intersection of \( Q \) with a surface of degree \( m + 4 - s \).

**Proof** Let \( h = m + 4 - s \). As in [32], Lemma III.2.4 p.63, applying the functor \( \text{Hom}_{\mathcal{O}_Q}(-, \mathcal{O}_Q) \) to the exact sequence

\[
0 \longrightarrow J_{C,Q}(h) \longrightarrow \mathcal{O}_Q(h) \longrightarrow \mathcal{O}_C(h) \longrightarrow 0
\]
we obtain (since $\omega_Q \cong O_Q(s - 4)$)
\[ 0 \to H^0 O_Q(-h) \to \text{Hom}(J_{C,Q}(h), O_Q) \to H^0 \omega_C(-m) \to 0. \]
Thus $\xi$ lifts to a nonzero morphism
\[ u : J_{C,Q}(h) \to O_Q. \]

If $u$ is injective, we define $C_0$ by $\text{Im} u = J_{C_0,Q}$ and conclude as in [32], Proposition III.2.3, that $C_0$ is either empty, in which case $C$ is the complete intersection of $Q$ with a surface of degree $h \geq 1$, or a curve, in which case $C$ is obtained from $C_0$ by an elementary biliaison of height $h$.

Suppose $u$ is not injective. The kernel of $u$ has the form $J_{D,Q}(h)$ for some subscheme $D$ of $Q$. Note that $D$ can’t be all of $Q$ because $u$ is not injective. On the other hand, $D$ must contain some component of $Q$: otherwise, the image of $u$, whose associated points are among those of $O_Q$, would be zero. Let $E$ be the subscheme of $Q$ whose ideal sheaf is the image of $u$, and let $F$ and $G$ be the two-dimensional components of $D$ and $E$ respectively. Looking at $u$ at the generic points of $Q$ we see that
\[ J_{G,Q} \cong O_F. \]
(geometrically, the surfaces $F$ and $G$ are linked by $Q$). Hence multiplication by the equation $f$ of $F$ kills $J_{G,Q} \supseteq J_{E,Q} = \text{Im}(u)$. It follows that $f \cdot \xi = 0$, which contradicts our hypothesis since $F$ is a proper subscheme of $Q$. Thus $u$ must be injective, and we are done.

**Remark 2.2.4** The Proposition is useful when $h = m + 4 - s \geq 1$ because it gives a criterion for the existence of an elementary biliaison of negative height $-h$ from $C$ to $C_0$. When $h \geq 1$, we have an isomorphism
\[ \text{Hom}(J_{C,Q}(h), O_Q) \cong H^0 \omega_C(-m) \]
and $u \in \text{Hom}(J_{C,Q}(h), O_Q)$ is injective (hence defines an elementary biliaison) if and only if the corresponding $\xi \in H^0 \omega_C(-m)$ satisfies the hypothesis of the Proposition.

**Remark 2.2.5** When $Q$ is integral, the hypothesis on $\xi$ is automatically satisfied, and the Proposition is well known. It is often used (see [32], Remark 2.7.b on page 65) in the following way: suppose $C$ is a curve in $\mathbb{P}^3$ which is contained in an integral surface $Q$ of degree $s$, and suppose $e(C) + 4 - s \geq 1$. Then picking a nonzero $\xi \in H^0 \omega_C(-e)$ we see that either $C$ is obtained by an elementary biliaison of height $e + 4 - s$ on $Q$, or $C$ is the complete intersection of $Q$ and a surface of degree $e + 4 - s$. In particular,
if $C$ is not a complete intersection, $C$ is not minimal in its biliaison class, and by Proposition 2.2.1 we have $\deg C > s \cdot (e + 4 - s)$, that is

$$e(C) < s + (\deg C)/s - 4.$$ 

This is a special case of the Speciality Theorem of Halphen-Gruson-Peskine which is proven in [15]: the argument on page 37 of [15] is essentially the one we have given. Later, while proving a version of the Speciality Theorem for general Cohen-Macaulay curves (Theorem 2.8.5 below), we will use the full strength of Proposition 2.2.3.

We give an example of how one can use Proposition 2.2.3 even when $Q$ is not integral by proving the following Proposition:

**Proposition 2.2.6** Let $Q$ be a surface of degree 2 in $\mathbb{P}^3$, and let $C \subseteq Q$ be a curve which is not a complete intersection and has spectrum $\{n^{h_C(n)}\}$. Then $C$ bilinks down on $Q$ if and only if $h_C(1) \geq 2$.

**Proof** One direction is easy: if $C$ bilinks down on $Q$, then by Proposition 2.2.1 the spectrum of $C$ contains $\{0, 1^2\}$ hence $h_C(1) \geq 2$.

Conversely, suppose that $h_C(1) \geq 2$. Then by Proposition 1.4.9 we have

$$H^0\omega_C(1) = \sum_{k \geq 1} k \cdot h_C(k) \geq 2.$$ 

Now if there is a linear form $h$ that kills all of $H^0\omega_C(1)$, then $h$ kills $H^0\omega_C(l)$ for all $l \leq 1$ because there is a linear form $x$ which is $\Omega_C$-regular. As in the proof of Step 7 in Theorem 1.7.1 we conclude that $h_C(l) = 1$ for all $1 \leq l \leq e(C) + 2$. In particular, $h_C(1) = 1$ which is a contradiction. Hence $h$ does not kill all of $H^0\omega_C(1)$, and therefore we can choose a nonzero element $\xi \in H^0\omega_C(1)$ such that $h \cdot \xi \neq 0$ for all linear forms $h$ dividing the equation of $Q$. Since $C$ is not a complete intersection, Proposition 2.2.3 implies that $C$ is obtained from some curve $C_0$ by an elementary biliaison on $Q$ of height $h = -1 + 4 - 2 = 1$, hence $C$ bilinks down on $Q$.

**Corollary 2.2.7** Let $Q$ be a surface of degree 2 in $\mathbb{P}^3$, and let $C \subseteq Q$ be a curve which is not ACM. Then $C \subseteq Q$ is minimal in the sense of [32], page 71, if and only if its spectrum satisfies $h_C(1) \leq 1$; and if $C$ is not minimal, it bilinks down on $Q$.

**Proof** Quite generally, if $C$ is not ACM and is not minimal, then according to [32] $C$ can be deformed with constant spectrum and Rao module to a
curve $D$ which bilinks down on some surface of degree at least two. By Proposition 2.2.1 we have

$$sp_C = sp_D \supseteq \{0, 1^2\}$$

hence $h_C(1) \geq 2$. Thus any non ACM curve with $h_C(1) \leq 1$ is minimal. The rest of the statement follows from Proposition 2.2.6.

2.3 Variation of the Spectrum under Liaison

In this section we give a formula for the variation of the spectrum under liaison: we refer to [26] for a general treatment of the theory of liaison. In contrast with the case of biliaison, the class of quasi ACM subschemes is not close under liaison, but the class of curves and the class of ACM subschemes are closed under liaison, so we can study these two cases.

For a curve $C \subseteq \mathbb{P}^N$, we define (following [32]) the Rao function $\rho_C$ of $C$ by the formula:

$$\rho_C(n) = \dim H^1(\mathbb{P}^N, J_C(n)).$$

Proposition 2.3.1

1. Let $C$ and $D$ be curves in $\mathbb{P}^N$, linked by the complete intersection $X$ of hypersurfaces of degrees $f_1, \ldots, f_{N-1}$. Then the spectra of $C$, $D$ and $X$ are related by the formula:

$$h_D(l) = h_X(l) - h_C(\sum f_i + 1 - N - l) + \partial^2 \rho_C(\sum f_i + 1 - N - l).$$

2. Let $V$ and $W$ be ACM subschemes of $\mathbb{P}^N$ of dimension $t$, linked by the complete intersection $X$ of hypersurfaces of degrees $f_1, \ldots, f_{N-1}$. Then the spectra of $V$, $W$ and $X$ are related by the formula:

$$h_W(l) = h_X(l) - h_V(\sum f_i + t - N - l).$$

We will not give the (easy) proof of this proposition, which we do not need in the sequel. We only remark that the proof of the first statement is based on the isomorphism

$$J_{D,X} \cong \omega_C(N + 1 - \sum f_i),$$

together with the equality $h^1 J_D(n) = h^1 J_C(\sum f_i - N - 1)$. 

2.4 Curvy Sheaves on $\mathbb{P}^3$.

In this and the following sections we are going to review the theory of the spectrum for a torsion free sheaf on $\mathbb{P}^3$ as developed in [42], but only in the special case of interest to us, that of “curvy sheaves” (to be defined below). These form a class of sheaves for which the spectrum behaves nicely and for which it is possible to define a principal spectrum. We also study the relation between the spectrum of a curvy sheaf of rank two and that of a curve which is the zero scheme of a section of the sheaf. The result will look like the biliaison formula of Proposition 2.2.1, so we may think that the curve is obtained by biliaison from the sheaf.

Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^3$. We define the singular locus $S(\mathcal{F})$ of $\mathcal{F}$ to be the set of points where $\mathcal{F}$ is not locally free. As $\mathcal{F}$ is coherent, $S(\mathcal{F})$ is a closed subspace of $\mathbb{P}^3$. Now consider a torsion free coherent sheaf $\mathcal{F}$ of rank $r$; then $\mathcal{F}$ is locally free in codimension $\leq 1$, so $S(\mathcal{F})$ has dimension at most one. If $L$ is any line not meeting $S(\mathcal{F})$, the restriction $\mathcal{F}_L$ of $\mathcal{F}$ to $L$ is locally free of rank $r$ over $\mathcal{O}_L$, hence it splits in the form

$$\mathcal{F}_L \cong \bigoplus_{i=1}^{r} \mathcal{O}_L(a_i)$$

for some integers $a_i$ satisfying $a_1 \geq \cdots \geq a_r$. The ordered sequence of integers

$$a_{\mathcal{F}}(L) = (a_1, \ldots, a_r)$$

is called the splitting type of $\mathcal{F}$ on $L$. One can show (see e.g. [41] Lemma 3.2.2 on page 54) that there is an open dense subset $U$ of the Grassmannian variety of lines in $\mathbb{P}^3$ such that the splitting type of $\mathcal{F}$ is the same for all lines in $U$, and it is minimal in the lexicographic order among the splitting types of $\mathcal{F}$ on lines not meeting $S(\mathcal{F})$; it is called the generic splitting type of $\mathcal{F}$. The importance of this notion for us results from Lemma 2.4.2 below, which is well-known (see [22] and [42]), but not usually stated explicitly. Before we proceed, we need a definition due to Hartshorne ([22] page 170).

**Definition 2.4.1** Let $R$ be the polynomial ring in four variables over $k$, let $V$ be a two dimensional subspace of the vector space $R_1$ and let $M$ be a finitely generated graded $R$-module. We say that $V$ is in general position for $M$ in degrees $\leq d$ if the following conditions are satisfied:

1. for any nonzero $x \in V$ and for any $n \leq d$, the multiplication map $x : M_{n-1} \rightarrow M_n$ is injective.
2. for any basis $(x, y)$ and for any $n \leq d$, the multiplication map $y : (M/xM)_{n-1} \rightarrow (M/xM)_n$ is injective.
Lemma 2.4.2 Assume \( \mathcal{F} \) is a torsion free coherent sheaf on \( \mathbb{P}^3 \) of homological dimension at most one. Let \( L \) be a line not meeting \( S(\mathcal{F}) \), let \( V \) be the two dimensional space of linear forms vanishing on \( L \), and let \((a_1, \ldots, a_r)\) be the splitting type of \( \mathcal{F} \) on \( L \). Then

1. \( V \) is in general position for \( H^1(\mathbb{P}^3, \mathcal{F}) \) in degrees \( \leq -a_1 - 1 \).

2. \( V \) is in general position for \((H^2(\mathbb{P}^3, \mathcal{F}))^* \) in degrees \( \leq a_r + 3 \).

Proof Recall that \( \mathcal{F} \) is of homological dimension at most one if it admits a resolution

\[
0 \rightarrow L_1 \rightarrow L_0 \rightarrow \mathcal{F} \rightarrow 0
\]

where \( L_1 \) and \( L_0 \) are locally free sheaves.

Let \( x \in V \) be a nonzero linear form vanishing on \( L \); \( x \) defines a plane \( H \) containing \( L \). I claim that the restriction \( \mathcal{F}_H \) of \( \mathcal{F} \) to \( H \) is torsion free of homological dimension at most one over \( \mathcal{O}_H \); indeed, as \( \mathcal{F} \) is torsion free, a locally free resolution

\[
0 \rightarrow L_1 \rightarrow L_0 \rightarrow \mathcal{F} \rightarrow 0
\]

of \( \mathcal{F} \) restricts to give a locally free resolution of \( \mathcal{F}_H \) over \( \mathcal{O}_H \). Now recall that a sheaf of homological dimension at most one on a regular integral scheme is torsion free if and only if it is locally free in codimension \( \leq 1 \).

On the other hand, \( S(\mathcal{F}) \cap H \) has codimension at most 2 in \( H \), because it doesn’t meet \( L \). Hence \( \mathcal{F}_H \) is torsion free of homological dimension at most one.

Next I claim that

\[
h^0 \mathcal{F}_H(l) = 0 \quad \text{if} \quad l \leq -a_1 - 1
\]

It is clear that \( h^0 \mathcal{F}_L(l) = 0 \) for \( l \leq -a_1 - 1 \) as \( \mathcal{F}_L \cong \bigoplus_{i=1}^r \mathcal{O}_L(a_i) \); to prove the claim for \( \mathcal{F}_H \), which we know is torsion free, we use the exact sequence

\[
0 \rightarrow \mathcal{F}_H(-1) \rightarrow \mathcal{F}_H \rightarrow \mathcal{F}_L \rightarrow 0
\]

which tells us that the function

\[
l \mapsto h^0 \mathcal{F}_H(l)
\]

is constant for \( l \leq -a_1 - 1 \). But for \( l \ll 0 \), \( h^0 \mathcal{F}_H(l) = 0 \) as \( \mathcal{F}_H \) has homological dimension at most one. This proves the claim. (Note that a similar argument shows \( h^0 \mathcal{F}(l) = 0 \) for \( l \leq -a_1 - 1 \).)

Now the kernel of multiplication by \( x : H^1(\mathcal{F}(l-1)) \rightarrow H^1(\mathcal{F}(l)) \) is a quotient of \( H^0 \mathcal{F}_H(l) \), which vanishes for \( l \leq -a_1 - 1 \); hence multiplication
by $x$ is injective in degrees $\leq -a_1 - 1$ which is the first half of the first statement in the Lemma.

Next we need to pick a basis $(x, y)$ of $V$: $x$ defines a plane $H$ as above, and $y$ is the equation of $L$ on $H$. We have an exact sequence

$$0 \rightarrow \mathcal{F}_H(-1) \xrightarrow{y} \mathcal{F}_H \rightarrow \mathcal{F}_L \rightarrow 0$$

from which we see, using the vanishing of $\mathcal{H}^0\mathcal{F}_L(l)$, that multiplication by $y : \mathcal{H}^1\mathcal{F}_H(l-1) \rightarrow \mathcal{H}^1\mathcal{F}_H(l)$ is injective for $l \leq -a_1 - 1$; as

$$\mathcal{H}^1\mathcal{F}(l)/x\mathcal{H}^1\mathcal{F}(l-1) \subseteq \mathcal{H}^1\mathcal{F}_H(l)$$

we conclude that $V$ is in general position for $\mathcal{H}^1\mathcal{F}$ in degrees $\leq -1 - a_1$.

The proof of the second statement is perfectly analogous; to simplify notation, we let

$$N = (\mathcal{H}^2(\mathbb{P}^3, \mathcal{F}))^*.$$ 

Recall that we give $N$ the grading $N_l = \mathcal{H}^2(\mathbb{P}^3, \mathcal{F}(-l))^*$. We will need the fact that

$$h^1\mathcal{F}_L(n) = h^2\mathcal{F}_H(n-1) = 0 \quad \text{if } n \geq -a_r - 1.$$ 

This is clear for $\mathcal{F}_L \cong \bigoplus_{i=1}^r \mathcal{O}_L(a_i)$; while for $\mathcal{F}_H$ it follows from the exact sequence

$$\mathcal{H}^1\mathcal{F}_L(n) \rightarrow \mathcal{H}^2\mathcal{F}_H(n-1) \rightarrow \mathcal{H}^2\mathcal{F}_H(n) \rightarrow 0.$$ 

together with Serre’s Vanishing Theorem which says $\mathcal{H}^2\mathcal{F}_H(n) = 0$ for $n \gg 0$.

We can now proceed to show that $V$ is in general position for $N$ in degrees $\leq a_r + 3$. Pick a basis $(x, y)$ of $V$: the kernel of multiplication by $x : N_{l-1} \rightarrow N_l$ is a quotient of $\mathcal{H}^2\mathcal{F}_H(1-l)^*$, which vanishes for $1 - l \geq -a_r - 2$, that is for $l \leq a_r + 3$.

Next we observe that $N_l/xN_{l-1}$ is a subspace of $\mathcal{H}^1\mathcal{F}_H(1-l)^*$, and that the kernel of multiplication by

$$y : \mathcal{H}^1\mathcal{F}_H(2-l)^* \rightarrow \mathcal{H}^1\mathcal{F}_H(1-l)^*$$

is a quotient of $\mathcal{H}^1\mathcal{F}_L(2-l)^*$, which vanishes for $2 - l \geq -a_r - 1$, that is for $l \leq a_r + 3$. This concludes the proof.

**Definition 2.4.3** We say that a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^3$ is *curvy* if it is torsion free of homological dimension at most one and the generic splitting type $(a_1, \ldots, a_r)$ of $\mathcal{F}$ satisfies

$$0 = a_1 \geq \ldots \geq a_r \geq -1.$$
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Remark 2.4.4 Thus a curvy sheaf is a torsion free sheaf which satisfies two special properties, namely it has homological dimension at most one and its splitting type is such that $a_1 - a_r \leq 1$, and furthermore is normalized so that $a_1 = 0$. The only integers occurring in the generic splitting type of a curvy sheaf are 0 and $-1$. Note that a reflexive sheaf has homological dimension at most one.

Lemma 2.4.5 Let $\mathcal{F}$ be a curvy sheaf on $\mathbb{P}^3$, let $L$ be a line on which the splitting type of $\mathcal{F}$ is generic, and let $H$ be a plane containing $L$. Then:

1. $H^0 \mathcal{F}_H(l) = H^0 \mathcal{F}(l) = 0$ for $l \leq -1$.
2. $H^2 \mathcal{F}_H(l) = H^3 \mathcal{F}(l - 1) = 0$ for $l \geq -1$

Proof Similar to that of Lemma 2.4.2.

Example 2.4.6 A rank one coherent sheaf is curvy if and only if it is isomorphic to the ideal sheaf of a curve, with the convention that a curve be a locally Cohen-Macaulay subscheme of $\mathbb{P}^3$ of pure dimension one, or the empty subscheme. Indeed, the ideal sheaf of a curve $C$ is torsion free of rank one, it has homological dimension one because a curve has no isolated or embedded points, and its generic splitting type is $(0)$, because for any line $L$ not meeting $C$ the ideal sheaf of $C$ restricted to $L$ is isomorphic to $\mathcal{O}_L$.

Conversely, if $\mathcal{F}$ is a curvy sheaf of rank one, then the generic splitting type of $\mathcal{F}$ is $(0)$, therefore $\mathcal{F}^{\vee\vee}$ is a reflexive rank one sheaf with $c_1 = 0$. Hence $\mathcal{F}^{\vee\vee} \cong \mathcal{O}$ and $\mathcal{F} \cong \mathcal{I}_X$ for some subscheme $X$ of $\mathbb{P}^3$. As $c_1(\mathcal{F}) = 0$, $X$ has dimension $\leq 1$, but $X$ cannot have zero dimensional components because $\mathcal{F}$ has homological dimension at most one. Thus $X$ is a curve.

Next we would like to characterize curvy sheaves in the class of rank two torsion free sheaves with homological dimension at most one. We will need the a version of the Theorem of Grauert-Mülich proven in [12], which gives us information about the generic splitting type of a semistable sheaf, but which fails in positive characteristic. Hence our result will be valid only in characteristic zero. We should remark that Hartshorne in [21] has been able to develop the theory of the spectrum for semistable reflexive sheaves of rank two without assuming characteristic zero.

Definition 2.4.7 Given a nonzero torsion free coherent sheaf $\mathcal{F}$ on $\mathbb{P}^3$, we define the slope of $\mathcal{F}$ to be the number

$$\mu(\mathcal{F}) = c_1(\mathcal{F})/\text{rank}\mathcal{F}. $$
We say that $\mathcal{F}$ is stable (respectively, semistable) if for all coherent sub-sheaves $\mathcal{E} \subseteq \mathcal{F}$ with $0 < \text{rank} \mathcal{E} < \text{rank} \mathcal{F}$ we have
\[ \mu(\mathcal{E}) < \mu(\mathcal{F}) \quad (\text{resp. } \mu(\mathcal{E}) \leq \mu(\mathcal{F})). \]
Note that two sheaves which are isomorphic on an open set containing all points of codimension $\leq 1$ have the same rank and the same first Chern class, hence the same slope. It follows that $\mathcal{F}$ is (semi)stable if and only if its double dual $\mathcal{F}^{\vee\vee}$ is (semi)stable, and one can show that this is also equivalent to $\mathcal{F}^{\vee}$ being (semi)stable (see [41] for details).

By definition, a torsion free coherent sheaf of rank one is stable. For rank two sheaves, there is a simple criterion which we now state. We say that a rank 2 torsion free sheaf $\mathcal{F}$ on $\mathbb{P}^3$ is normalized if $c_1(\mathcal{F})$ is either 0 or $-1$. For a general rank 2 torsion free sheaf $\mathcal{F}$, there is a unique twist $\mathcal{F}_{\text{Norm}} = \mathcal{F}(l)$ which is normalized: this follows from the formula $c_1(\mathcal{F}(l)) = c_1(\mathcal{F}) + 2l$.

**Proposition 2.4.8** Let $\mathcal{F}$ be a normalized torsion free coherent sheaf of rank two on $\mathbb{P}^3$.

1. Suppose $c_1(\mathcal{F}) = 0$. Then $\mathcal{F}$ is stable if and only if $\text{H}^0(\mathbb{P}^3, \mathcal{F}^{\vee\vee}) = 0$; $\mathcal{F}$ is semistable if and only if $\text{H}^0(\mathbb{P}^3, \mathcal{F}^{\vee\vee}(-1)) = 0$.

2. Suppose $c_1(\mathcal{F}) = -1$. Then $\mathcal{F}$ is stable if and only if it is semistable if and only if $\text{H}^0(\mathbb{P}^3, \mathcal{F}^{\vee\vee}) = 0$.

For the proof, note that we may assume that $\mathcal{F}$ is reflexive, and then we can use [41], Lemma II.1.2.5. or [21], Lemma 3.1. Now we state the form of the Theorem of Grauert-Mülich (see [12]) we will need. The characteristic zero hypothesis in this theorem is essential: see [9].

**Theorem 2.4.9 (Grauert-Mülich)** Assume the ground field $k$ has characteristic zero. Let $\mathcal{F}$ be a torsion free coherent sheaf of rank two on $\mathbb{P}^3$, which is normalized so that $c_1(\mathcal{F}) = 0$ or $-1$. Assume that
\[ \text{H}^0(\mathbb{P}^3, \mathcal{F}^{\vee\vee}(-1)) = 0. \]
Then for a general line $L$ we have
\[ \mathcal{F}_L \cong \mathcal{O}_L \oplus \mathcal{O}_L(c_1). \]

**Proof** Note that if $c_1 = 0$ or if $c_1 = -1$ and $\text{H}^0(\mathbb{P}^3, \mathcal{F}^{\vee\vee}) = 0$, $\mathcal{F}$ is semistable and our statement is a special case of the Corollary to Theorem 4.2 in [12]. In the other case we need to consider, we have $c_1 = -1$, \[ \text{H}^0(\mathbb{P}^3, \mathcal{F}^{\vee\vee}) = 0, \]
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\[ H^0(\mathcal{F}^{\vee\vee}(-1)) = 0 \] and a nonzero section \( s \in H^0(\mathcal{F}^{\vee\vee}) \). Then \( s \) defines an exact sequence:

\[ 0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{F}^{\vee\vee} \longrightarrow \mathcal{J}_Y(-1) \longrightarrow 0 \]

where \( Y \) is a curve in \( \mathbb{P}^3 \) (since \( \mathcal{F}^{\vee\vee} \) is reflexive and \( s \) is a section of minimal degree). Then restricting the above exact sequence to a line \( L \) not meeting \( Y \cup S(\mathcal{F}) \), we see \( \mathcal{F}_L \cong \mathcal{F}^{\vee\vee}_L \cong \mathcal{O}_L \oplus \mathcal{O}_L(-1) \).

**Corollary 2.4.10** Assume the ground field \( k \) has characteristic zero. A rank two torsion free coherent sheaf \( \mathcal{F} \) of homological dimension at most one is curvy if and only if it is normalized and \( H^0(\mathbb{P}^3, \mathcal{F}^{\vee\vee}(-1)) = 0 \).

**Proof** Assume that \( \mathcal{F} \) is a rank two torsion free coherent sheaf of homological dimension at most one; if \( \mathcal{F} \) is normalized and \( H^0(\mathbb{P}^3, \mathcal{F}^{\vee\vee}(-1)) = 0 \), then the generic splitting type of \( \mathcal{F} \) is \((0, c_1)\) by the Theorem of Grauert-Müllich, hence \( \mathcal{F} \) is curvy. On the other hand, if \( \mathcal{F} \) is curvy, its generic splitting type is \((0, a_2)\) with \( 0 \geq a_2 = c_1(\mathcal{F}) \geq -1 \). Hence \( \mathcal{F} \) is normalized. Furthermore, \( \mathcal{F}^{\vee\vee} \) is also curvy, because its generic splitting type is the same as that of \( \mathcal{F} \). So \( H^0(\mathbb{P}^3, \mathcal{F}^{\vee\vee}(-1)) = 0 \) by Lemma 2.4.5.

### 2.5 The Spectrum of a Curvy Sheaf

In this section we define the spectrum for a curvy sheaf on \( \mathbb{P}^3 \). The classic references on the theory of the spectrum for classes of torsion free sheaves on \( \mathbb{P}^3 \) are [5, 21, 42]; our definition is in fact a special case of that in [42]. Roughly speaking, the spectrum counts the number of generators in low degrees of \( H^1(\mathbb{P}^3, \mathcal{F}) \) (resp. \( H^2(\mathbb{P}^3, \mathcal{F}^*) \)) over the ring \( S_L \cong k[x, y] \), where \( L \) is a general line.

**Proposition 2.5.1** Let \( \mathcal{F} \) be a sheaf on \( \mathbb{P}^3 \), and suppose that \( \mathcal{F} \) is either a curvy sheaf, or a rank 2 reflexive sheaf with \( c_1(\mathcal{F}) = 0 \) or \(-1\) satisfying \( H^0(\mathcal{F}(-1)) = 0 \). There exists a unique set of integers with multiplicities \( sp_\mathcal{F} = \{n^{h_\mathcal{F}(n)}\} \) such that

\[
 h_\mathcal{F}(n) = \begin{cases} 
 \partial^2 h^1 \mathcal{F}(n) & \text{if } n \leq -1 \\
 \partial^2 h^2 \mathcal{F}(n) & \text{if } n \geq 0 
\end{cases}
\]

**Proof** If \( \mathcal{F} \) is a rank 2 reflexive sheaf with \( c_1(\mathcal{F}) = 0 \) or \(-1\) satisfying \( H^0(\mathcal{F}(-1)) = 0 \), the statement follows from Theorem 7.1 page 151 of [21]. Note that such a sheaf in characteristic zero is curvy by Corollary 2.4.10.
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Now suppose that $\mathcal{F}$ is curvy. We need to show that $\partial^2 h^1 \mathcal{F}(n) \geq 0$ if $n \leq -1$ (resp. that $\partial^2 h^2 \mathcal{F}(n) \geq 0$ if $n \geq 0$). Pick a line $L$ such that the splitting type of $\mathcal{F}$ on $L$ is $(0, a_2, \ldots, a_r)$ with $a_r \geq -1$. By Lemma 2.4.2 the vector space $V$ of linear forms vanishing on $L$ is in general position for $H^1_*(\mathcal{F})$ in degrees $\leq -1$ and for $H^2_*(\mathcal{F})$ in degrees $\leq a_r + 3$. It follows that $\partial^2 h^1(\mathcal{F}(n)) \geq 0$ for $n \leq -1$ and, if we let $g(n) = h^2 \mathcal{F}(-n)$, that $\partial^2 g(n) \geq 0$ for $n \geq 0$. But

$$\partial^2 g(n) = \partial^2 [h^2 \mathcal{F}(l)](2 - n)$$

and we conclude that $\partial^2 h^2 \mathcal{F}(n) \geq 0$ for $n \geq -1 - a_r$, and in particular for $n \geq 0$. This finishes the proof.

**Definition 2.5.2** If $\mathcal{F}$ is a curvy sheaf, or a rank 2 reflexive sheaf with $c_1(\mathcal{F}) = 0$ or $-1$ satisfying $H^0(\mathcal{F}(-1)) = 0$, we define the spectrum of $\mathcal{F}$ to be the sequence of integers $\text{sp}_e \mathcal{F}$ in the above Proposition.

**Remark 2.5.3** Note that our definition differs from the classical one of [21] and [42]: if $h'_e(\mathcal{F})$ denotes the multiplicity with which the integer $n$ appears in the spectrum as defined in [21] or in [42], we have $h'_e(\mathcal{F}) = h_\mathcal{F}(-1 - n)$ (and our notation for the splitting type of $\mathcal{F}$ also differs from [42]). This modification is necessary to make the definition consistent with that of the spectrum of a curve. Indeed, if $\mathcal{F}$ is the ideal sheaf of a curve, then the spectrum of $\mathcal{F}$ as we have defined it coincide with the spectrum of $\mathcal{C}$ introduced in the previous chapter. To see this, we need only to observe that if $\mathcal{C}$ is a curve, then

$$h_\mathcal{C}(n) = \partial^2 h^0 \mathcal{C}(n) = \begin{cases} \partial^2 h^1 \mathcal{C}(n) & \text{if } n \leq -1 \\ \partial^2 h^2 \mathcal{C}(n) & \text{if } n \geq 0 \end{cases}$$

The following Proposition is a special case of [42], Propositions 2.6 and 2.7.

**Proposition 2.5.4** Let $\mathcal{F}$ be a curvy sheaf on $\mathbb{P}^3$ with spectrum $\text{sp}_e \mathcal{F} = \{n h_\mathcal{F}(n)\}$. Then

$$\sum_{n \in \mathbb{Z}} h_\mathcal{F}(n) = -\chi \mathcal{F}(-1) + \chi \mathcal{F}(-2)$$

$$\sum_{n \in \mathbb{Z}} n h_\mathcal{F}(n) = \chi \mathcal{F}(-1)$$
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Proof

We have

\[ \sum h_F(n) = \sum_{n \leq -1} h_F(n) + \sum_{n \geq 0} h_F(n) \]

\[ = \sum_{n \leq -1} \partial^2 h_1 F(n) + \sum_{n \geq 0} \partial^2 h_2 F(n) \]

\[ = \partial h_1 (F)(-1) + \partial h_2 (F)(-1). \]

Now pick a plane \( H \) containing a line \( L \) on which the splitting type of \( F \) is generic; by Lemma 2.4.5 we have \( h^0 H(-1) = h^2 H(-1) = 0 \). Using the long exact sequence of cohomology associated to

\[ 0 \longrightarrow F(-2) \longrightarrow F(-1) \longrightarrow H(-1) \longrightarrow 0 \]

we see that

\[ \partial h_1 (F)(-1) + \partial h_2 (F)(-1) = h^1 H(-1) = -\chi H(-1). \]

This gives us the first formula because

\[ -\chi H(-1) = -\chi F(-1) + \chi F(-2). \]

To prove the second formula, observe that if \( g : \mathbb{Z} \rightarrow \mathbb{Z} \) is a numerical function, then integration by parts gives \( g(-1) = \sum_{n \geq 0} n \partial^2 g(n) \) if \( g(n) = 0 \) for \( n \gg 0 \), while \( g(-1) = -\sum_{n \leq -1} n \partial^2 g(n) \) if \( g(n) = 0 \) for \( n \ll 0 \). Since \( H^0 F(-1) = H^1 F(-1) = 0 \) by Lemma 2.4.5 we have:

\[ \chi F(-1) = h^2 (F)(-1) - h^1 (F)(-1) \]

\[ = \sum_{n \geq 0} nh_F(n) - (\sum_{n \leq -1} nh_F(n)) \]

\[ = \sum_{n} nh_F(n) \]

and we are done.

Definition 2.5.5 Let \( F \) be a curvy sheaf on \( \mathbb{P}^3 \). We call the number \(-\chi F(-1) + \chi F(-2)\) the curvy degree of \( F \) and we denote it by the symbol \( d(F) \).

Remark 2.5.6 By Proposition 2.5.4 the spectrum of \( F \) consists of \( d(F) \) integers. If \( F \) is the ideal sheaf of a curve \( C \), then \( d(F) \) is the degree of \( C \), and the second formula in 2.5.4 is equivalent to the formula for the arithmetic genus we gave in Chapter one. If \( F \) is curvy of rank two, its first
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Chern class is either 0 or $-1$ and by Riemann-Roch we have $d(F) = c_2(F)$ and

$$\chi_F(-1) = -\frac{1}{2}c_2(2 + c_1) + \frac{1}{2}c_3$$

If $F$ is a rank 2 reflexive sheaf with $c_1(F) = 0$ or $-1$ satisfying $H^0(F(-1)) = 0$, it is still true that $\sum_{n \in \mathbb{Z}} h_F(n) = c_2(F)$ and

$$\sum_{n \in \mathbb{Z}} nh_F(n) = -\frac{1}{2}c_2(2 + c_1) + \frac{1}{2}c_3 :$$

this was proven by Hartshorne ([21], Theorem 7.1 and Proposition 7.3).

We are now going to recall some properties of the spectrum established in [21] and in [42]. The following Theorem is due to Hartshorne:

**Theorem 2.5.7 ([21], Theorem 7.5)** Let $F$ be a reflexive rank 2 sheaf on $\mathbb{P}^3$ with $c_1 = 0$ or $-1$ and suppose $H^0(F(-1)) = 0$. Let $\{n h_F(n)\}$ be the spectrum of $F$. Then

1. If for some integer $l \leq -2$ we have $h_F(l) > 0$, then $h_F(n) > 0$ for $l \leq n \leq -2$.

2. If for some $l \geq -c_1$ we have $h_F(l) > 0$, then $h_F(n) > 0$ for $-c_1 \leq n \leq l$.

Furthermore, if $H^0(F) = 0$ (i.e., $F$ is stable), then

3. If for some integer $l \leq -2$ we have $h_F(l) > 0$, then $h_F(n) > 0$ for $l \leq n \leq -1$.

4. Suppose for some $l \geq 0$ we have $h_F(l) > 0$. Then $h_F(n) > 0$ for $0 \leq n \leq l$, and, if $c_1 = 0$, either $h_F(-1) > 0$ or $h_F(0) \geq 2$.

For more general sheaves, a similar Theorem is proven in [42]. We will state this result for curvy sheaves. To this end, we need to introduce more notation. For a sheaf $F$ on $\mathbb{P}^3$ and a plane $H$ we define

$$\sigma(F, H) = \dim H^0(H, \text{Ext}^1_{\mathcal{O}_H}(F_H, \mathcal{O}_H))$$

If $F$ is reflexive, then for a general plane $H$ the restriction $F_H$ is locally free on $H$, hence $\sigma(F, H) = 0$.

We can now state the Theorem. Its proof can be found in [42]: the first two statements are special cases of [42] Proposition 2.4 , while the third one is contained in [42] Theorem 4.1. The proof is characteristic free. Our Theorem 1.7.1 was modeled after this result and its predecessors in [5, 21, 22].
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**Theorem 2.5.8** Let $F$ be a curvy sheaf on $\mathbb{P}^3$ with spectrum $\text{sp}_F = \{n^{h_F(n)}\}$ and generic splitting type $\{0, \ldots, a_r\}$ (so that $a_r = 0$ or $-1$).

1. Let $H$ be a general plane with respect to $F$. If for some integer $l \leq -2$ we have
   \[ \sum_{n \leq l} h_F(n) > \sigma(F, H) \]
   then $h_F(n) > 0$ for $l \leq n \leq -2$.

2. If for some $l \geq -a_r$ we have $h_F(l) > 0$, then $h_F(n) > 0$ for $-a_r \leq n \leq l$.

3. Let $b = b(F)$ be the largest integer $n$ for which $h_F(n) > 0$. If for some integer $l$ with $-a_r < l < b$ we have $h_F(l) = 1$, then $h_F(n) = 1$ for $l \leq n \leq b$.

**Remark 2.5.9** If $F$ is the ideal sheaf of a curve $C$, we have $a_r = 0$ and $b(F) = e(C) + 2$. Hence the last two statements of the Theorem follow in this case from Theorem 1.7.1. On the other hand, we did not prove an analogue of the first statement for quasi ACM subschemes.

### 2.6 The Correspondence in $\mathbb{P}^3$ Between Curves and Rank Two Sheaves

In this section we study the relation between the spectrum of a rank two curvy sheaf $F$ and that of a curve which is the zero scheme of a section of $F(n)$. The correspondence between rank two reflexive sheaves and curves which are generically local complete intersection has been extensively studied by Hartshorne in [21, 22, 25]. There is an analogous correspondence between rank two torsion free sheaves of homological dimension at most one and (quasi ACM) curves. In particular, the following two Propositions are the analogue of Theorem 4.1 and Proposition 4.2 in [21].

**Proposition 2.6.1** Let $c_1$ be equal to 0 or $-1$ and let $n$ be an integer. There is a one-to-one correspondence between:

1. pairs $(F, s)$ where $F$ is a rank two torsion free sheaf on $\mathbb{P}^3$ of homological dimension at most one with $c_1(F) = c_1$, and $s \in H^0 F(n)$ is a global section whose zero set has codimension at least two (up to isomorphisms of $F$).

2. pairs $(C, \xi)$ where $C$ is a curve in $\mathbb{P}^3$ (possibly empty) and $\xi \in H^0(C, \omega_C(4-2n-c_1))$. 

Furthermore, \( s \) has a nonempty zero set if and only if \( \xi \) is nonzero.

**Proof** Given \((F, s)\) as in the statement, we have an exact sequence
\[
0 \to \mathcal{O}_{\mathbb{P}^3}(-n) \xrightarrow{s} F \to T \to 0
\]
for some coherent sheaf \( T \). We claim that \( T \) is torsion free. Since \( F \) has homological dimension at most one, \( T \) also has homological dimension at most one, hence it is torsion free if and only if it is locally free in codimension one. Now \( \mathcal{O}_{\mathbb{P}^3} \) and \( F \) are locally free in codimension one, and by assumption \( s(x) \in F_x/m_xF_x \) is nonzero if \( x \in \mathbb{P}^3 \) has codimension one. Hence \( T \) is locally free in codimension one, therefore it is torsion free of rank one and of homological dimension at most one. Now by Example 2.4.6 there is a curve \( C \) such that \( T \cong \mathcal{J}_C(c_1(T)) \). We have \( c_1(T) = n + c_1 \), so we can rewrite the above exact sequence as:
\[
0 \to \mathcal{O}_{\mathbb{P}^3}(-n) \xrightarrow{s} F \to \mathcal{J}_C(n + c_1) \to 0 \tag{2.5}
\]
This corresponds to an element
\[
\xi \in \text{Ext}^1(\mathcal{J}_C(n + c_1), \mathcal{O}_{\mathbb{P}^3}(-n)) \cong H^0(C, \omega_C(4 - 2n - c_1)).
\]
Note that \( \xi = 0 \) if and only if the corresponding extension splits if and only if the zero set of \( s \) is empty.

Conversely, given a pair \((C, \xi)\) as in the statement, \( \xi \) corresponds to an extension (2.5), which defines \( F \) and \( s \). Since \( C \) is locally Cohen-Macaulay, as in Example 2.4.6 we see that \( \mathcal{J}_C \) is torsion free of homological dimension at most one. Therefore \( F \) is torsion free of rank two and of homological dimension at most one. Finally, \( s \) can vanish only along \( C \), hence its zero set has codimension at least two.

**Proposition 2.6.2** Let \( c_1 \) be equal to 0 or \(-1\), and let \( n \) be an integer. Let \( C \) be a curve in \( \mathbb{P}^3 \), fix \( \xi \in H^0(C, \omega_C(4 - 2n - c_1)) \) and consider the sheaf \( F \) corresponding to \((C, \xi)\) as in Proposition 2.6.1. Then

1. \( H^0 F^\vee(-1) = 0 \) if and only if \( n \geq 0 \) and the map
\[
H^0 \mathcal{O}_{\mathbb{P}^3}(n + c_1 - 1) \longrightarrow H^0 \omega_C(4 - n - 1)
\]
which sends \( f \) to \( f\xi \) is injective; in particular, \( H^0 F^\vee(-1) = 0 \) if \( n \geq 0 \) and no component of \( C \) lies on a surface of degree \( n + c_1 - 1 \).

2. \( H^0 F^\vee = 0 \) if and only if \( n \geq 1 \) and the map
\[
H^0 \mathcal{O}_{\mathbb{P}^3}(n + c_1) \longrightarrow H^0 \omega_C(4 - n)
\]
which sends \( f \) to \( f\xi \) is injective; in particular, \( H^0 F^\vee = 0 \) if \( n \geq 1 \) and no component of \( C \) lies on a surface of degree \( n + c_1 \).
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Proof We only prove the first statement since the proof of the second one is perfectly analogous. We need to observe that $\mathcal{F}^{\vee\vee} \cong \mathcal{F}^{\vee}(c_1)$ as $\mathcal{F}^{\vee}$ is a rank two reflexive sheaf with first Chern class $-c_1$. We apply the functor $\text{Hom}(-, \mathcal{O}_{\mathbb{P}^3}(c_1-1))$ to the exact sequence 2.5 to get

$$
0 \longrightarrow H^0\mathcal{O}(-1-n) \longrightarrow H^0\mathcal{F}^{\vee}(c_1-1) \cong H^0\mathcal{F}^{\vee\vee}(-1) \longrightarrow H^0\mathcal{O}(n+c_1-1)
$$

Thus $H^0\mathcal{F}^{\vee\vee}(-1) = 0$ if and only if $n \geq 0$ and the map $\phi$ is injective, and if we identify $\text{Ext}^1(\mathcal{J}_C(n+c_1), \mathcal{O}(c_1-1))$ with $H^0\omega_C(4-n-1)$, the map $\phi$ sends $f$ to $f\xi$. Finally, if $C$ has no component contained in a surface of degree $n + c_1 - 1$, $\xi$ cannot be killed by any form of degree $n + c_1 - 1$ because $\Omega_C = H^0(C, \omega_C)$ has the same associated points as $C$.

Corollary 2.6.3 Let $c_1$ be equal to 0 or $-1$, and let $n$ be an integer. Let $C$ be a curve in $\mathbb{P}^3$, fix $\xi \in H^0(C, \omega_C(4-2n-c_1))$ and consider the sheaf $\mathcal{F}$ corresponding to $(C, \xi)$ as in Proposition 2.6.1. If the ground field has characteristic zero, then $\mathcal{F}$ is curvy if and only if $n \geq 0$ and the map $H^0\mathcal{O}_{\mathbb{P}^3}(n+c_1-1) \longrightarrow H^0\omega_C(4-n-1)$ which sends $f$ to $f\xi$ is injective.

Proof By Corollary 2.4.10, in characteristic zero a rank two torsion sheaf $\mathcal{F}$ of homological dimension at most one is curvy if and only if it is normalized and $H^0(\mathbb{P}^3, \mathcal{F}^{\vee\vee}(-1)) = 0$.

The next Proposition describes the relation between the spectra of $\mathcal{F}$ and $C$.

Proposition 2.6.4 Let $c_1$ be equal to 0 or $-1$, and let $n$ be an integer. Let $C$ be a curve in $\mathbb{P}^3$, fix $\xi \in H^0(C, \omega_C(4-2n-c_1))$ and consider the sheaf $\mathcal{F}$ corresponding to $(C, \xi)$ as in Proposition 2.6.1. Suppose that $\mathcal{F}$ is either a curvy sheaf or a rank 2 reflexive sheaf with $c_1(\mathcal{F}) = 0$ or $-1$ satisfying $H^0(\mathcal{F}(-1)) = 0$. Then

$$
\text{sp}_C = \text{sp}_X(-c_1-n) \cup \text{sp}_X
$$

where $X$ is a complete intersection of two surfaces of degrees $n$ and $n+c_1$ (if $n+c_1 \leq 0$, set $\text{sp}_X = \emptyset$).

Proof In any case the spectrum of $\mathcal{F}$ is well defined and $H^0\mathcal{F}^{\vee\vee}(-1) = 0$, so that $n \geq 0$ by Proposition 2.6.2. By Proposition 1.5.3 the spectrum of
CHAPTER 2. THE SPECTRUM OF A CURVE IN $\mathbb{P}^3$

the complete intersection $X$ of two surfaces of degrees $n+c_1$ and $n$ is given by the formula:

$$h_X(l+n+c_1) = \begin{cases} 
\max (0, n-l-1) & \text{if } l \geq 0 \\
\max (0, l+n+c_1+1) & \text{if } l \leq -1 
\end{cases}$$

(if $n+c_1 \leq 0$, this gives $h_X(n) = 0$ for all $n$).

Now fix $l \geq -2$. By Lemma 2.4.5 or by Serre's duality we have

$$H^3(\mathbb{P}^3, F(l)) = 0 \quad \text{if } l \geq -2.$$  

Thus for $l \geq -2$ we have an exact sequence

$$0 \longrightarrow H^2F(l) \longrightarrow H^2J_C(l+n+c_1) \longrightarrow H^3O(l-n) \longrightarrow 0$$

from which we deduce that for $l \geq 0$

$$h_C(l+n+c_1) = \partial^2 h^2 J_C(l+n+c_1) = \partial^2 h^2 F(l) + \max (0, n-l-1) = h_F(l) + h_X(l+n+c_1)$$

which proves half of our claim.

On the other hand, we have $h^0 J_C(l+n+c_1) = 0$ for $l \leq -1$ because

$$h^0 F(l) = h^0 F^\vee (l) = 0 \quad \text{for } l \leq -1.$$  

Hence for $l \leq -1$ we have

$$h_F(l) = \partial^2 h^1 F(l) = \partial^2 h^1 J_C(l+n+c_1) = \partial^2 h^0 O_C(l+n+c_1) - \partial^2 h^0 O_{\mathbb{P}^3}(l+n+c_1) = h_C(l+n+c_1) - \max (0, l+n+c_1+1) = h_C(l+n+c_1) - h_X(l+n+c_1).$$

Summing up we have shown that for all integers $l$ we have

$$h_C(l+n+c_1) = h_F(l) + h_X(l+n+c_1),$$

which means

$$\text{sp}_C = \text{sp}_F(-c_1-n) \cup \text{sp}_X.$$

**Remark 2.6.5** Later on we will show that, if $F$ is curvy, then $\text{sp}_X$ is contained in the principal spectrum of $C$ relative to a line $L$ such that $F_L \cong O_L \oplus O_L(c_1)$. 

2.7 The Principal Spectrum of a Curvy Sheaf

In this section we use the approach of [5] to the theory of the spectrum for sheaves to prove the following result: if $F$ is a curvy sheaf on $\mathbb{P}^3$ with spectrum $\{n^{h,\nu(n)}\}$ and $L$ is a line on which $F$ attains its generic splitting type, there is a surjective morphism of $S_L$-modules:

$$\bigoplus_{n \in \mathbb{Z}} S_L(-n)^{h,\nu(n)} \rightarrow H^1_*(\mathbb{P}^3, F).$$

This is the analogue in the theory of sheaves of the surjective map $A_C \rightarrow M_C$ that comes naturally with a curve $C$, and allows us to define the principal spectrum of a curvy sheaf. For us the crucial fact is that, if a rank 2 curvy sheaf $F$ corresponds to a curve $C$ as in Proposition 2.6.1, the principal spectrum of $F$ and $C$ are related in the expected way.

The main idea of [5] is that, when we look at the spectrum, we consider the action of linear forms vanishing on a general line $L$ on modules like $A_C$ or $H^1_\nu F$, so that geometrically we are looking at the projection from $L$ to the projective line parametrizing the planes through $L$; this projection is only defined on $\mathbb{P}^3 \setminus L$, but it can be extended to the blow up $B$ of $\mathbb{P}^3$ along $L$; the resulting map $q : B \rightarrow \mathbb{P}^1$ is a $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ and will be the center of our study.

So fix a line $L$ in $\mathbb{P}^3$ and let $B$ be the blow up of $\mathbb{P}^3$ along $L$. $B$ comes with two maps:

$$\begin{array}{c}
B \xrightarrow{p} \mathbb{P}^3 \\
\downarrow q \\
\mathbb{P}^1
\end{array}$$

where $p$ is the blow up map and $q$ is the extension of the projection from $L$.

We can describe $B$ geometrically as follows: let $\lambda \rightarrow H_\lambda$ be a parametrization of the projective line of planes through $L$; then

$$B = \{(P, \lambda) \in \mathbb{P}^3 \times \mathbb{P}^1 : P \in H_\lambda\}$$

and $p$ and $q$ are the restrictions to $B$ of the projections of $\mathbb{P}^3 \times \mathbb{P}^1$ onto its factors; note that $p$ induces an isomorphism

$$H_\lambda \cong \lambda \times H_\lambda = q^{-1}(\lambda)$$

between a plane through $L$ and its strict transform in $B$. We fix homogeneous coordinates $t, u$ on $\mathbb{P}^1$, and $x, y, z, w$ on $\mathbb{P}^3$, so that the line $L$ has equation $x = y = 0$. Then $B \subseteq \mathbb{P}^3 \times \mathbb{P}^1$ has equation $xu - yt = 0$.

The exceptional divisor $E = p^{-1}(L)$ is a trivial $\mathbb{P}^1$ sub-bundle of $B$, hence isomorphic to $L \times \mathbb{P}^1$. We let $O_B(k, l) = p^*O_{\mathbb{P}^3}(k) \otimes q^*O_{\mathbb{P}^1}(l)$; the
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Picard group of \( B \) is isomorphic to \( \mathbb{Z} \oplus \mathbb{Z} \) with generators \( \mathcal{O}_B(1, 0) \) and \( \mathcal{O}_B(0, 1) \). If \( \mathcal{G} \) is a sheaf on \( B \), we will write \( \mathcal{G}(k, l) \) for the sheaf \( \mathcal{G} \otimes \mathcal{O}_B(k, l) \).

Note that an equation for \( E \) is given by:

\[
f = x/t = y/u \in H^0(B, \mathcal{O}_B(1, -1))
\]

Hence \( \mathcal{O}_B(1, -1) \) is isomorphic to the invertible sheaf \( \mathcal{O}_B(E) \) corresponding to the divisor \( E \). This can be seen more geometrically as follows: if \( H \) is a plane passing through \( L \), the global transform \( p^*(H) \) is the divisor \( E + H \) on \( B \), hence \( \mathcal{O}_B(E + H) \cong \mathcal{O}_B(1, 0) \); on the other hand, \( H \) is a fiber of \( q \), so \( \mathcal{O}_B(H) \cong \mathcal{O}_B(0, 1) \), and therefore \( \mathcal{O}_B(E) \cong \mathcal{O}_B(1, -1) \).

**Lemma 2.7.1** Let \( \mathcal{F} \) be a coherent sheaf on \( \mathbb{P}^3 \) and assume that \( \mathcal{F} \) is locally free on \( L \). Then the natural map \( \mathcal{F} \to p_* p^* \mathcal{F} \) is an isomorphism, and

\[
p^* : H^i(\mathbb{P}^3, \mathcal{F}) \to H^i(B, p^* \mathcal{F})
\]

is an isomorphism for all integers \( i \).

**Proof** It is enough to prove the first statement for \( \mathcal{F} = \mathcal{O}_{\mathbb{P}^3} \), since the claim is local on \( \mathbb{P}^3 \), \( \mathcal{F} \) is locally free on \( L \) and \( p : B \setminus E \to \mathbb{P}^3 \setminus L \) is an isomorphism. So we only need to show that the natural map \( \mathcal{O}_{\mathbb{P}^3} \to p_* p^* \mathcal{O}_{\mathbb{P}^3} \) is an isomorphism, and this follows from the fact that \( p \) is a projective birational morphism and \( \mathbb{P}^3 \) is normal.

To prove the second statement, we observe that \( R^i p_* p^* \mathcal{F} = 0 \) for \( i > 0 \): as above we may assume that \( \mathcal{F} = \mathcal{O}_{\mathbb{P}^3} \), and the higher direct images of \( \mathcal{O}_B \cong p^* \mathcal{O}_{\mathbb{P}^3} \) vanish as \( p \) is the blow up of a smooth subvariety of \( \mathbb{P}^3 \). Now the Leray’s spectral sequence degenerates to give isomorphisms

\[
p_* : H^i(B, p^* \mathcal{F}) \to H^i(\mathbb{P}^3, \mathcal{F})
\]

for all integers \( i \).

Now we turn our attention to the map \( q : B \to \mathbb{P}^1 \). The restriction of the sheaf \( \mathcal{O}_B(1, 0) \) to a fiber \( H \) of \( q \) is \( \mathcal{O}_H(1) \), and \( \mathcal{O}_B(1, 0) \) is the \( \mathcal{O}(1) \) sheaf associated to the bundle map \( q : B \to \mathbb{P}^1 \); hence \( B \) is isomorphic to the projective bundle \( \mathbb{P}(\mathcal{V}) \), where \( \mathcal{V} = q_* \mathcal{O}_B(1, 0) \) is a rank three invertible sheaf on \( \mathbb{P}^3 \). The Lemma that follows describes the sheaves \( q_* \mathcal{O}_B(n, 0) \) for \( n \geq 0 \). We will denote by \( E_n \) the closed subscheme of \( B \) corresponding to the effective divisor \( nE \), that is, the subscheme defined by \( f^n = 0 \).

**Lemma 2.7.2** For each \( n \geq 0 \) there is a split exact sequence of locally free sheaves on \( \mathbb{P}^1 \):

\[
0 \to q_* \mathcal{O}_B(0, n) \xrightarrow{f^n} q_* \mathcal{O}_B(n, 0) \to q_* \mathcal{O}_{E_n}(n, 0) \to 0.
\]
Furthermore, we have $q_* \mathcal{O}_B(0, n) \cong \mathcal{O}_{P^1}(n)$ and

$$S^n \mathcal{V} \cong q_* \mathcal{O}_B(n, 0) \cong \bigoplus_{j=0}^{n} \mathcal{O}_{P^1}(j)^{n-j+1}.$$ 

**Proof** For each integer $n \geq 0$ we have an exact sequence:

$$0 \to \mathcal{O}_B(-n, n) \xrightarrow{f^n} \mathcal{O}_B \to \mathcal{O}_{E^n} \to 0.$$ 

We twist by $\mathcal{O}_B(n, 0)$ and apply $q_*$. Note that $R^1q_* \mathcal{O}_B(0, n)$ vanishes: by the projection formula we have $R^1q_* \mathcal{O}_B(0, n) \cong R^1q_* \mathcal{O}_B \otimes \mathcal{O}_{P^1}(n)$ and this is zero by the theorem of cohomology and base change, since the restriction of $\mathcal{O}_B$ to a fiber $H$ of $q$ is $\mathcal{O}_H$ whose first cohomology group vanishes. Thus we obtain the exact sequence:

$$0 \to q_* \mathcal{O}_B(0, n) \to q_* \mathcal{O}_B(n, 0) \to q_* \mathcal{O}_{E^n}(n, 0) \to 0. \quad \text{(2.6)}$$

Next we observe that $q_* \mathcal{O}_B \cong \mathcal{O}_{P^1}$ because $q$ is a bundle map, hence $q_* \mathcal{O}_B(0, n) \cong q_* \mathcal{O}_B \otimes \mathcal{O}_{P^1}(n) \cong \mathcal{O}_{P^1}(n)$ for all integers $n$. The Künneth formula for $E$ gives

$$q_* \mathcal{O}_E(1, 0) \cong H^0(L, \mathcal{O}_L(1)) \otimes \mathcal{O}_{P^1} \cong \mathcal{O}_{P^1} \oplus \mathcal{O}_{P^1}.$$ 

Since $\text{Ext}^1_{\mathcal{O}_P}(\mathcal{O}_{P^1}, \mathcal{O}_{P^1}(1)) = 0$, it follows that the exact sequence (2.6) splits when $n = 1$. Hence

$$\mathcal{V} = q_* \mathcal{O}_B(1, 0) \cong \mathcal{O}_{P^1}(1) \oplus \mathcal{O}_{P^1} \oplus \mathcal{O}_{P^1}.$$ 

Now we can compute

$$S^n \mathcal{V} \cong \bigoplus_{j=0}^{n} H^0(L, \mathcal{O}_L(n-j)) \otimes \mathcal{O}_{P^1}(j) \cong \bigoplus_{j=0}^{n} \mathcal{O}_{P^1}(j)^{n-j+1}$$

and this is isomorphic to $q_* \mathcal{O}_B(n, 0)$ by [20], Proposition II.7.11. As $q_* \mathcal{O}_B(0, n) \cong \mathcal{O}_{P^1}(n)$, the first map in the exact sequence (2.6) must have as image the $\mathcal{O}_{P^1}(n)$ direct summand of $q_* \mathcal{O}_B(n, 0)$, hence the sequence is split exact.

Next we consider the global sections of these sheaves; we denote by $S = k[t, u]$ the coordinate ring of $P^1$ and by $R = k[x, y, z, w]$ the coordinate ring of $P^3$, where $x$ and $y$ are equations for the line $L$; by Lemma 2.7.1 we have an isomorphism of graded rings

$$p^* : R \xrightarrow{\bigoplus_{n \in \mathbb{Z}} \text{H}^0(P^3, \mathcal{O}_{P^3}(n))} \bigoplus_{n \in \mathbb{Z}} \text{H}^0(B, \mathcal{O}_B(n, 0));$$
by Lemma 2.7.2 $S = \bigoplus H^0(\mathbb{P}^1, O_{\mathbb{P}^1}(n))$ is isomorphic to $\bigoplus H^0(B, O_B(0, n))$; hence the first map in the exact sequence (2.6) gives rise to an injective map of $k$-algebras $\iota : S \to R$. Of course, $\iota$ is the map of graded rings corresponding to the projection $q : \mathbb{P}^3 \setminus L \to \mathbb{P}^1$: the image of $\iota$ is the sub-algebra $k[x, y]$ generated by the linear forms vanishing on $L$. Indeed, $\iota$ on forms of degree $n$ is multiplication by $f^n$, and in particular we have $\iota(t) = x$ and $\iota(u) = y$.

Now, given a sheaf $G$ on $B$, the vector space

$$\bigoplus_{n \in \mathbb{Z}} H^i(B, G(0, n))$$

has a natural structure of $S$-module; the vector space

$$\bigoplus_{n \in \mathbb{Z}} H^i(B, G(n, 0))$$

has a natural structure of $R$-module and hence (via $\iota$) of $S$-module.

We can now state the main result of this section:

**Theorem 2.7.3** Let $\mathcal{F}$ be a curvy sheaf on $\mathbb{P}^3$, and let $L$ be a line not meeting the singular locus of $\mathcal{F}$, on which $\mathcal{F}$ has its generic splitting type. Let $B \overset{p}{\to} \mathbb{P}^3$ the blow up of $L$. The sheaf $\mathcal{H} = R^1q_*p^*\mathcal{F}(-1) \otimes O_{\mathbb{P}^1}(1)$ is locally free of rank $d(\mathcal{F})$ on $\mathbb{P}^1$; more precisely we have

$$\mathcal{H} \cong \bigoplus_{n \in \mathbb{Z}} O_{\mathbb{P}^1}(-n)^{h_{\mathcal{F}}(n)}$$

where $\{n^{h_{\mathcal{F}}(n)}\}$ is the spectrum of $\mathcal{F}$. Furthermore:

1. There is a surjective homogeneous morphism of graded $S$-modules

$$H^i_*(\mathbb{P}^1, \mathcal{H}) \to H^1_*(\mathbb{P}^3, \mathcal{F})$$

which is an isomorphism in degrees $\leq -1$.

2. There is an injective homogeneous morphism of graded $S$-modules

$$H^2_*(\mathbb{P}^3, \mathcal{F}) \to H^1_*(\mathbb{P}^1, \mathcal{H})$$

which is an isomorphism in degrees $\geq -2$.

Before we can prove the Theorem, we need a technical Lemma.
Lemma 2.7.4 Let $\mathcal{G}$ be a coherent sheaf on $B$ which is flat over $\mathbb{P}^1$. Assume that $\mathcal{G}$ is locally free on $E$ and that

$$\mathcal{G}_E \cong \bigoplus_{i=1}^{r} \mathcal{O}_E(a_i, 0)$$

where each $a_i$ is equal to either $-2$ or $-1$. For all integers $n \geq 0$ we have

1. $q_*(\mathcal{G} \otimes \mathcal{O}_E(n, 0)) = 0$ for $l \leq 0$.
2. $R^1q_*(\mathcal{G} \otimes \mathcal{O}_E(n, 0)) = 0$ for $l \geq n$.
3. $R^2q_*(\mathcal{G} \otimes \mathcal{O}_E(n, 0)) = 0$ for all integers $l$.
4. $H^1(\mathbb{P}^1, q_*(\mathcal{G} \otimes \mathcal{O}_E(n, 0))) = 0$ for $l \geq n$.
5. $H^1(B, \mathcal{G} \otimes \mathcal{O}_E(n, 0)) = 0$ for $l \geq n$.
6. $H^1(B, \mathcal{G} \otimes \mathcal{O}_E(0, l)) = 0$ for $l \leq -1$.
7. $H^2(B, \mathcal{G} \otimes \mathcal{O}_E(n, 0)) = 0$ for $l \geq n$.

Proof We prove the Lemma by induction on $n$. The initial case $n = 0$ is trivial because $\mathcal{O}_{E_0} = 0$. Now look at the exact sequence

$$(2.7) \quad 0 \rightarrow \mathcal{O}_E(-n, n) \rightarrow \mathcal{O}_{E_{n+1}} \rightarrow \mathcal{O}_{E_n} \rightarrow 0$$

We tensor with $\mathcal{G}(n+k, 0)$: the sequence remains exact because $\mathcal{G}$ is locally free on (a neighborhood of) $E$, so we get

$$(2.8) \rightarrow \mathcal{G} \otimes \mathcal{O}_E(k, n) \rightarrow \mathcal{G} \otimes \mathcal{O}_{E_{n+1}}(n+k, 0) \rightarrow \mathcal{G} \otimes \mathcal{O}_{E_n}(n+k, 0) \rightarrow 0.$$ 

We first prove statement 1: by the Künneth formula we have

$$q_*(\mathcal{G} \otimes \mathcal{O}_E(k, n)) \cong \bigoplus_i q_*(\mathcal{O}_E(k+a_i, n)) \cong \bigoplus_i H^0(L, \mathcal{O}_L(k+a_i)) \otimes \mathcal{O}_{\mathbb{P}^1}(n)$$

which is zero if $k+a_i \leq -1$ for all $i$ and in particular if $k \leq 0$. Thus if we apply $q_*$ to the exact sequence (2.8) we see that $q_*(\mathcal{G} \otimes \mathcal{O}_{E_{n+1}}(n+k, 0))$ is a subsheaf of $q_*(\mathcal{G} \otimes \mathcal{O}_{E_n}(n+k, 0))$ for $k \leq 0$, that is, $q_*(\mathcal{G} \otimes \mathcal{O}_{E_{n+1}}(l, 0))$ is a subsheaf of $q_*(\mathcal{G} \otimes \mathcal{O}_{E_n}(l, 0))$ for $l \leq n$. Thus if we have $q_*(\mathcal{G} \otimes \mathcal{O}_{E_n}(l, 0)) = 0$ for $l \leq 0 \leq n$, then $q_*(\mathcal{G} \otimes \mathcal{O}_{E_{n+1}}(l, 0))$ also vanishes for $l \leq 0$. By induction this proves statement 1.

We now prove statement 3: we have

$$R^i q_*(\mathcal{G} \otimes \mathcal{O}_E(k, n)) \cong \bigoplus_j R^i q_*(\mathcal{O}_E(k+a_j, n)) = 0$$
for all \( i \geq 2 \) and all \( k \) because the fibers of \( q : E \to \mathbb{P}^1 \) are one dimensional. Hence applying \( q_* \) to the exact sequence (2.8) we find for each integer \( k \) an isomorphism

\[
R^2 q_*(G \otimes O_{E_{n+1}}(n + k, 0)) \cong R^2 q_*(G \otimes O_{E_n}(n + k, 0)).
\]

Since \( k \) is arbitrary, this proves by induction \( R^2 q_*(G \otimes O_{E_n}(l, 0)) = 0 \) for all \( l \). Next we prove statement 2. By the K"unneth formula we have

\[
R^1 q_* O_E(l, n) \cong H^1(L, O_L(l)) \otimes O_{\mathbb{P}^1}(n),
\]

which vanishes if \( l \geq -1 \). Hence

\[
R^1 q_* G \otimes O_E(k, n) \cong \bigoplus R^1 q_* O_E(k + a_i, n) = 0 \quad \text{for all } k \geq 1
\]
as each \( a_i \) is greater or equal than \(-2\). Since we know that \( R^2 q_* G \otimes O_E(k, n) = 0 \) for all \( k \), from the exact sequence (2.8) we deduce

\[
R^1 q_*(G \otimes O_{E_{n+1}}(n + k, 0)) \cong R^1 q_*(G \otimes O_{E_n}(n + k, 0)) \quad \text{for } k \geq 1.
\]

By induction this proves \( R^1 q_*(G \otimes O_{E_{n+1}}(l, 0)) = 0 \) for \( l \geq n + 1 \), which is statement 2. To prove 4, we use again the fact that \( R^1 q_* G \otimes O_E(k, n) = 0 \) for \( k \geq 1 \). Applying \( q_* \) to the exact sequence (2.8) we get

\[
0 \to q_* G \otimes O_E(k, n) \to q_* G \otimes O_{E_{n+1}}(n + k, 0) \to q_* G \otimes O_{E_n}(n + k, 0) \to 0.
\]

Now by the K"unneth formula

\[
H^1(\mathbb{P}^1, q_* G \otimes O_E(k, n)) = H^2(\mathbb{P}^1, q_* G \otimes O_E(k, n)) = 0
\]
because \( k + a_i \geq -1 \) and \( n \geq 0 \). Hence we have isomorphisms:

\[
H^1(\mathbb{P}^1, q_* G \otimes O_{E_{n+1}}(n + k, 0)) \cong H^1(\mathbb{P}^1, q_* G \otimes O_{E_n}(n + k, 0))
\]

for all \( k \geq 1 \). By induction statement 4 follows as above. Statement 5 is a consequence of 2 and 4: indeed, for any coherent sheaf \( \mathcal{N} \) on \( B \) in the Leray’s spectral sequence

\[
H^i(\mathbb{P}^1, R^j q_* \mathcal{N}) \Rightarrow H^{i+j}(B, \mathcal{N})
\]

all the differentials are zero, so that for \( i + j = 1 \) we have an exact sequence:

\[
0 \to H^1(\mathbb{P}^1, q_* \mathcal{N}) \to H^1(B, \mathcal{N}) \to H^0(\mathbb{P}^1, R^1 q_* \mathcal{N}) \to 0.
\]

Now taking \( \mathcal{N} \) to be the sheaf \( G \otimes O_{E_n}(l, 0) \) we see that 5 follows from 2 and 4. To prove 6, we use once more the Leray’s Spectral Sequence; since \( q_* G \otimes O_{E_n}(0, l) \cong q_* (G \otimes O_{E_n}) \otimes O_{\mathbb{P}^1}(l) \) vanishes by 1, it is enough to show that

\[
H^0(\mathbb{P}^1, R^1 q_* G \otimes O_{E_n}(0, l)) = 0
\]
for \( l \leq -1 \). This follows by induction from the vanishing of \( H^0(\mathbb{P}^1, R^l q_* \mathcal{G} \otimes \mathcal{O}_E(-n, l)) \) for \( l \leq -1 \) which in turn follows from the isomorphisms \( R^l q_* \mathcal{G} \otimes \mathcal{O}_E(-n, l) \cong \oplus R^l q_* \mathcal{O}_E(a_i - n, l) \cong \oplus H^1(L, \mathcal{O}_L(a_i - n)) \otimes \mathcal{O}_{\mathbb{P}^1}(l) \).

Statement 7 follows from the Leray’s Spectral Sequence together with 2 and 3.

**Proof of Theorem 2.7.3** We consider the sheaf \( p^*(\mathcal{F}(-1)) \) on \( B \): it is flat on \( \mathbb{P}^1 \) since its restriction to a fiber \( H \) of \( q \) is isomorphic to \( \mathcal{F}_H(-1) \), whose Hilbert polynomial is independent of \( H \). Furthermore, the \( H^0 \) and \( H^2 \) modules of \( \mathcal{F}_H(-1) \) vanish because \( \mathcal{F} \) is curvy. It follows from the Theorem of Cohomology and Base Change that the sheaf \( \mathcal{H}(-1) = R^l q_* p^* \mathcal{F}(-1) \) is locally free on \( \mathbb{P}^1 \) of rank \( d(\mathcal{F}) = h^1(H, \mathcal{F}_H(-1)) \), while \( R^0 q_* p^* \mathcal{F}(-1) = R^2 q_* p^* \mathcal{F}(-1) = 0 \).

We now prove 1. We’d like to show that we can work on \( B \): we have seen that we may identify \( \bigoplus_n H^1(B, \mathcal{O}_B(0, n)) \) with the coordinate ring \( S = k[t, u] \) of \( \mathbb{P}^1 \) and \( \bigoplus_n H^1(B, \mathcal{O}_B(n, 0)) \) with the coordinate ring \( R = k[x, y, z, w] \) of \( \mathbb{P}^3 \). Then by Lemma 2.7.1 there is an isomorphism of \( R \)-modules

\[
H^1(\mathbb{P}^3, \mathcal{F}(-1)) \cong \bigoplus_n H^1(B, p^* \mathcal{F}(n - 1, 0)).
\]

On the other hand, since \( R^0 q_* p^* \mathcal{F}(-1) = 0 \), the Leray’s Spectral Sequence

\[
H^i(\mathbb{P}^1, R^l q_* p^* \mathcal{F}(-1)) \Rightarrow H^{i+l}(B, p^* \mathcal{F}(-1))
\]

gives an isomorphism of \( S \)-modules

\[
\bigoplus_n H^1(B, p^* \mathcal{F}(-1, n)) \cong \bigoplus_n H^0(\mathbb{P}^1, \mathcal{H}(n - 1)).
\]

Thus to prove 1 we have to show that there is a surjective homogeneous morphism of graded \( S \)-modules

\[
\phi : \bigoplus_n H^1(B, p^* \mathcal{F}(-1, n)) \twoheadrightarrow \bigoplus_n H^1(B, p^* \mathcal{F}(n - 1, 0))
\]

which is an isomorphism in degrees \( n \leq 0 \). We will define \( \phi \) in degree \( n \) to be multiplication by \( f^n \) for all integers \( n \), but we will need to show that this makes sense for \( n \) negative.
For \(n \geq 0\) we have an exact sequence
\[
0 \to \mathcal{O}_B(0, n) \xrightarrow{f_n} \mathcal{O}_B(n, 0) \to \mathcal{O}_{E_n}(n, 0) \to 0.
\]
(2.9)

Tensoring with \(p^*\mathcal{F}(-1)\) and taking cohomology we obtain a \(k\)-linear map
\[
\phi_n : H^1(B, p^*\mathcal{F}(-1, n)) \to H^1(B, p^*\mathcal{F}(n - 1, 0))
\]
for each \(n \geq 0\). Note that \(\phi_0\) is the identity map on \(H^1(B, p^*\mathcal{F}(-1, 0))\). We claim that \(\phi_n\) is surjective for \(n \geq 0\). We need to use Lemma 2.7.4 with \(G = p^*\mathcal{F}(-1)\): the hypotheses of the Lemma are satisfied because \(\mathcal{F}(-1)\) is locally free on \(L\) and \(\mathcal{F}_L(-1)\) splits as a direct sum of copies of \(\mathcal{O}_L(-1)\) and \(\mathcal{O}_L(-2)\). We look at:
\[
0 \to p^*\mathcal{F}(-1, n) \to p^*\mathcal{F}(n, -1) \to p^*\mathcal{F}(-1) \otimes \mathcal{O}_{E_n}(n, 0) \to 0
\]
Taking cohomology we get an exact sequence:
\[
H^1(B, p^*\mathcal{F}(-1, n)) \xrightarrow{\phi_n} H^1(B, p^*\mathcal{F}(n - 1, 0)) \to H^1(B, p^*\mathcal{F}(-1) \otimes \mathcal{O}_{E_n}(n, 0)).
\]
The last cohomology group vanishes by Lemma 2.7.4, and we conclude that \(\phi_n\) is surjective for all \(n \geq 0\).

Now we turn to negative degrees. For each \(n \leq 0\) we have an exact sequence:
\[
0 \to \mathcal{O}_B(n, 0) \xrightarrow{f_n} \mathcal{O}_B(0, n) \to \mathcal{O}_{E_{-n}}(0, n) \to 0.
\]
(2.10)

Tensoring with \(p^*\mathcal{F}(-1)\) and taking cohomology we obtain maps (going the “wrong” way) \(\theta_n : H^1(B, p^*\mathcal{F}(n - 1, 0)) \to H^1(B, p^*\mathcal{F}(-1, n))\). For all \(n \leq 0\) we have \(H^0(B, p^*\mathcal{F}(-1) \otimes \mathcal{O}_{E_{-n}}(0, n)) = 0\) because
\[
q_* p^*\mathcal{F}(-1) \otimes \mathcal{O}_{E_{-n}}(0, n) \cong q_* (p^*\mathcal{F}(-1) \otimes \mathcal{O}_{E_{-n}}) \otimes \mathcal{O}_{\mathbb{P}^1}(n) = 0
\]
by Lemma 2.7.4, and \(H^1(B, p^*\mathcal{F}(-1) \otimes \mathcal{O}_{E_{-n}}(0, n)) = 0\), again by Lemma 2.7.4. It follows that \(\theta_n\) is an isomorphism for all \(n \leq 0\) and we can define \(\phi_n = \theta_n^{-1}\) for \(n \leq 0\). Note that this is consistent with our previous definition for \(n = 0\) as \(\phi_0\) and \(\theta_0\) are the identity map on \(H^1(B, p^*\mathcal{F}(-1))\).

Now we can define
\[
\phi = \bigoplus_n \phi_n : \bigoplus_n H^1(B, p^*\mathcal{F}(-1, n)) \to \bigoplus_n H^1(B, p^*\mathcal{F}(n - 1, 0)).
\]
We have already seen that \(\phi\) is an isomorphism in degrees \(\leq 0\) and that it is surjective; it remains to show that \(\phi\) is \(S\)-linear. It is enough to prove
that for every $\alpha \in H(B, p^* F(-1, n))$ and every linear form $l \in S$ we have $\phi_{n+1}(l\alpha) = \iota(l)\phi_n(\alpha)$, where $\iota$ denotes the inclusion $S \to R$ induced by $\mathcal{O}_S(0, n) \to \mathcal{O}_B(n, 0)$. This follows from the fact that by construction $\phi_n$ is multiplication by $f^n$ for all integers $n$, while $\iota$ on linear forms is simply multiplication by $f$.

The proof of 2 is similar. By Lemma 2.7.1 we have an isomorphism

$$H^2(\mathbb{P}^3, F(n - 1)) \cong H^2(B, p^* F(n - 1, 0)).$$

On the other hand, since $R^2 q_* p^* F(-1) = 0$, the Leray’s Spectral Sequence

$$H^i(\mathbb{P}^1, R^j q_* p^* F(-1)) \Rightarrow H^{i+j}(B, p^* F(-1))$$

gives an isomorphism

$$H^2(B, p^* F(-1, n)) \cong H^1(\mathbb{P}^1, H(n - 1)).$$

Thus to prove 2 we have to show that there is an injective homogeneous morphism of graded $S$-modules

$$\psi : \bigoplus_n H^2(B, p^* F(n - 1, 0)) \to \bigoplus_n H^2(B, p^* F(-1, n))$$

which is an isomorphism in degrees $n \geq -1$. Now for each $n \geq 0$ the exact sequence (2.9) induces a $k$-linear map

$$\zeta_n : H^2(B, p^* F(-1, n)) \to H^2(B, p^* F(n - 1, 0)).$$

By Lemma 2.7.4 we have

$$H^1(B, p^* F(-1) \otimes \mathcal{O}_{\mathbb{P}^1}(n, 0)) = H^2(B, p^* F(-1) \otimes \mathcal{O}_{\mathbb{P}^1}(n, 0)) = 0$$

hence $\zeta_n$ is an isomorphism for $n \geq 0$, and we can define $\psi_n = \zeta_n^{-1}$ for $n \leq 0$.

Now we turn to negative degrees. For each $n \leq 0$ the exact sequence 2.10 defines a map $\psi_n : H^2(B, p^* F(n - 1, 0)) \to H^2(B, p^* F(-1, n))$. Note that this is consistent with our previous definition for $n = 0$ as $\psi_0$ and $\zeta_0$ are the identity map on $H^2(B, p^* F(-1))$. We still have to show that $\psi_n$ is injective for all $n \leq -1$ and an isomorphism for $n = -1$. Injectiveness follows from the vanishing of $H^1(B, p^* F(-1) \otimes \mathcal{O}_{\mathbb{P}^1}(n, 0))$ which can be proven using the Leray’s Spectral Sequence and Lemma 2.7.4. To prove that $\psi_{-1}$ is surjective as well, it is enough to show that $H^2(B, p^* F(-1) \otimes \mathcal{O}_{\mathbb{P}^1}(0, -1)) = 0$. Now we have $R^2 q_* p^* F(-1) \otimes \mathcal{O}_{\mathbb{P}^1}(0, -1) = 0$ by Lemma 2.7.4, while

$$R^1 q_* p^* F(-1) \otimes \mathcal{O}_{\mathbb{P}^1}(0, -1) \cong \bigoplus_i H^1(E, \mathcal{O}_{a_i}) \otimes \mathcal{O}_{p^*}(1).$$
Hence in the exact sequence

\[ 0 \to H^1(\mathbb{P}^1, R^1 q_* p^* \mathcal{F}(-1) \otimes \mathcal{O}_E(0, -1)) \to H^2(B, p^* \mathcal{F}(-1) \otimes \mathcal{O}_E(0, -1)) \to H^2(\mathbb{P}^1, R^2 q_* p^* \mathcal{F}(-1) \otimes \mathcal{O}_E(0, -1)) \to 0 \]

all groups are zero, and we conclude that \( \psi_{-1} \) is an isomorphism, so that the map \( \psi = \oplus_n \psi_n \) has all desired properties.

Finally, we have seen that \( H^i \) is locally free of finite rank on \( \mathbb{P}^1 \), so we can write

\[ H \cong \bigoplus_{n \in \mathbb{Z}} O_{\mathbb{P}^1}(-n)^{b_n}; \]

we claim that \( b_n = h_{\mathcal{F}}(n) \). By 1 we have \( h^1(\mathbb{P}^3, \mathcal{F}(n)) = h^0(\mathbb{P}^1, H(n)) \) for \( n \leq -1 \). Taking second differences we see that

\[ h_{\mathcal{F}}(n) = b_n \]

for \( n \leq -1 \); on the other hand, by 2 we have \( h^2(\mathbb{P}^3, \mathcal{F}(n)) = h^1(\mathbb{P}^1, H(n)) \) for \( n \geq -2 \), which implies

\[ h_{\mathcal{F}}(n) = b_n \]

for \( n \geq 0 \). This concludes the proof.

Looking back at the proof of the theorem we see that we can say something about the kernel of the map \( \phi \); specifically, there is an exact sequence of \( S \)-modules:

\[ 0 \to H^0(\mathbb{P}^3, \mathcal{F}) \to B(\mathcal{F}, L) \to H^0(\mathbb{P}^1, H) \xrightarrow{\phi} H^1(\mathbb{P}^3, \mathcal{F}) \to 0 \]

where

\[ B(\mathcal{F}, L) = \bigoplus_{n \geq 0} H^0(B, p^* \mathcal{F} \otimes \mathcal{O}_{E_{n+1}}(n, 0)). \]

This sequence is the analogue for sheaves of the standard sequence

\[ 0 \to I_C \to R \to A_C \to M_C \to 0 \]

for curves \( C \).

**Proposition 2.7.5** \( B(\mathcal{F}, L) \) and \( H^0(L, \mathcal{F}_L) \otimes_k S \) are isomorphic as graded \( k \)-vector spaces.

**Proof** The statement should be that they are isomorphic as \( S \)-modules, but we content ourselves with proving our weaker statement. We have to show that the Hilbert functions \( b(n) \) of \( B(\mathcal{F}, L) \) and \( a(n) \) of \( H^0(L, \mathcal{F}_L) \otimes_k S \)
are the same. For \( n \leq -1 \), we have \( b(n) = 0 \) by definition, and \( a(n) = 0 \) because in the splitting type of \( \mathcal{F} \) over \( L \) only 0 and \(-1\) occur. Therefore it is enough to show that

\[
\partial^2 a(n) = \partial^2 b(n) \quad \text{for} \quad n \geq 0.
\]

For \( n \geq 0 \), we have \( \partial^2 a(n) = h^0(L, \mathcal{F}_L(n)) \). To compute \( \partial^2 b(n) \) we use (2.11): we note that for \( n \geq 0 \) we have

\[
\partial^2 h^0(\mathbb{P}^1, \mathcal{H}(n)) = \partial^2 h^2(\mathbb{P}^3, \mathcal{F}(n)),
\]

while \( h^3(\mathbb{P}^3, \mathcal{F}(n)) = 0 \) for \( n \geq -2 \); now it follows from (2.11) that

\[
\partial^2 b(n) = \partial^2 \chi(n) = \chi_{\mathcal{F}_L}(n)
\]

for \( n \geq 0 \). Since \( \chi_{\mathcal{F}_L}(n) = h^0(L, \mathcal{F}_L(n)) \) for \( n \geq 0 \), we are done.

**Corollary 2.7.6** Let \( \mathcal{F} \) be a curvy sheaf on \( \mathbb{P}^3 \) with spectrum \( \{ h^0(\mathcal{F}, n) \} \). If \( H^0(\mathbb{P}^3, \mathcal{F}) = 0 \) and \( h_{\mathcal{F}}(n) = 0 \) for \( n \leq -1 \), then

\[
h_{\mathcal{F}}(0) \geq h^0(L, \mathcal{F}_L).
\]

**Proof** Look at the exact sequence (2.11). According to our hypothesis, all modules are zero in negative degree, and the statement follows by counting dimensions in degree zero.

Let \( \mathcal{F} \) be a sheaf on \( \mathbb{P}^3 \) and let \( L \) be a line. We let

\[
\mu_{\mathcal{F}, L}(n) = \dim_k (M_{\mathcal{F}} \otimes \mathcal{S}_L) \kappa_n,
\]

where \( M_{\mathcal{F}} = H^1(\mathbb{P}^3, \mathcal{F}) \): \( \mu_{\mathcal{F}, L}(n) \) is the number of minimal generators of \( M_{\mathcal{F}} \) over \( \mathcal{S}_L \) in degree \( n \). If \( \mathcal{F} \) is curvy and \( L \) is a line on which \( \mathcal{F} \) has its generic splitting type, then by Theorem 2.7.3 we have

\[
h_{\mathcal{F}, L}(n) \overset{\text{def}}{=} h_{\mathcal{F}}(n) - \mu_{\mathcal{F}, L}(n) \geq 0.
\]

**Definition 2.7.7** Let \( \mathcal{F} \) be a curvy sheaf on \( \mathbb{P}^3 \) and let \( L \) be a line on which \( \mathcal{F} \) attains its generic splitting type. We call the set of integers with multiplicities

\[
\text{sp}_{\mathcal{F}, L} = \{ n h_{\mathcal{F}, L}(n) \}
\]

the principal spectrum of \( \mathcal{F} \) relative to the line \( L \).
Remark 2.7.8 The sequence $h_{\mathcal{F},L}(n)$ is the Hilbert function of the map $P(\mathcal{F}, L)$ of the map

$$B(\mathcal{F}, L) \otimes_S k \to H^0(\mathbb{P}^1, \mathcal{H}) \otimes_S k$$

obtained from (2.11) upon tensoring with $k$.

Remark 2.7.9 In contrast with the case of curves, the principal spectrum of a curvy sheaf can be empty: this is the case, for example, for instanton vector bundles of rank two on $\mathbb{P}^3$ with $c_1 = 0$. Their spectrum is $\{-1^{c_2}\}$, and in this case $\phi$ gives a minimal free presentation of $M_{\mathcal{F}}$ over $S$.

Remark 2.7.10 One can prove as we did for quasi ACM subschemes that there is a an open dense subset $U$ of the Grassmannian variety of lines in $\mathbb{P}^3$ such that the principal spectra of $\mathcal{F}$ relative to lines in $U$ all coincide. We define the principal spectrum of $\mathcal{F}$ to be $sp_{\mathcal{F},L}$ for $L \in U$.

Proposition 2.7.11 Let $C$ be a curve in $\mathbb{P}^3$, let $\mathcal{F}$ be a rank 2 curvy sheaf on $\mathbb{P}^3$, and suppose $C$ and $\mathcal{F}$ correspond to each other as in Proposition 2.6.1, so that there is an exact sequence:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-n) \to \mathcal{F} \to \mathcal{J}_C(n + c_1) \to 0,$$

where $c_1 = c_1(\mathcal{F})$ is either $-1$ or $0$. Let $L$ be a line not meeting $C$ or the singular locus of $\mathcal{F}$, on which $\mathcal{F}$ attains its generic splitting type. Then

$$sp_{C,L} = sp_{\mathcal{F},L}(-n - c_1) \cup sp_X$$

where $X$ is the complete intersection of two surfaces of degrees $n + c_1$ and $n$.

Proof By Proposition 2.6.4 we have for all integers $l$:

$$(2.12) \quad h_C(l) = h_{\mathcal{F}}(l - c_1 - n) + h_X(l).$$

From the exact sequence in the statement we see that $M_C \cong M_{\mathcal{F}}(-c_1 - n)$, hence

$$\mu_{C,L}(l) = \mu_{\mathcal{F},L}(l - c_1 - n).$$

Thus subtracting $\mu_{C,L}(l)$ from the two sides in equation 2.12 we obtain the desired formula.
2.8 More Properties of the Spectrum. The Speciality Theorem

In this section we use all the machinery developed so far in this chapter to prove Theorem 2.8.2 below, which strengthens and generalizes Theorem 1.7.1. As a Corollary, we obtain a version of the Speciality Theorem of Halphen-Gruson-Peskine ([15], Theorem 1.2).

Let $C$ be a curve in $\mathbb{P}^3$ with index of speciality $e = e(C)$. As usual, we denote by $s(C)$ the smallest degree of a surface containing $C$.

Given $\xi \in H^0 \omega_C(-e)$, we define $\Sigma(\xi)$ to be the smallest integer $n$ such that there is a nonzero form $g$ on $\mathbb{P}^3$ of degree $n$ which annihilates $\xi$. Note that $\Sigma(\xi) \leq s(C)$, and we let $\Sigma(C)$ be the largest of the integers $\Sigma(\xi)$ as $\xi$ varies in $H^0 \omega_C(-e)$. We always have $\Sigma(C) \leq s(C)$, with equality holding if $C$ is an integral curve. The following Lemma gives some geometric meaning to $\Sigma(C)$.

Lemma 2.8.1 Let $C$ be a curve in $\mathbb{P}^3$. Then

1. For every curve $D \subseteq C$ with $e(D) = e(C)$, we have $\Sigma(D) \leq \Sigma(C)$.
2. There exists a curve $D \subseteq C$ such that $e(D) = e(C)$ and $s(D) = \Sigma(D) = \Sigma(C)$.
3. If there exists a curve $D \subseteq C$ with $e(D) = e(C)$ such that for every curve $E \subseteq D$ with $e(E) = e(D)$ we have $s(E) \geq t$, then $\Sigma(C) \geq t$.

Proof First we note that if $D$ is a curve contained in $C$, then $\Omega_D$ is a submodule of $\Omega_C$, so that statement 1 follows immediately from the definition of $\Sigma$. To prove 2, pick $\xi \in H^0 \omega_C(-e) \setminus \{0\}$ which is killed by a nonzero form $f$ of degree $\Sigma(C)$ but not by any form of smaller degree. Let $F$ be the surface defined by $f$; since $f$ kills $\xi$, $C \cap F$ has dimension one. Let $D$ be the largest curve contained in $C \cap F$: $D$ is obtained from $C \cap F$ by removing all isolated and embedded points. Since $D$ and $C \cap F$ have dimension one and differ only in dimension zero, their Grothendieck dualizing sheaves are isomorphic:

$$\omega_D \cong \omega_{C \cap F}.$$

Thus from the exact sequence

$$\mathcal{O}_C(-\Sigma) \xrightarrow{f} \mathcal{O}_C \to \mathcal{O}_{C \cap F} \to 0$$

we obtain

$$0 \to \Omega_D \to \Omega_C \xrightarrow{f} \Omega_C(\Sigma).$$
As \( f \) kills \( \xi \), we see that \( \xi \) lives in the submodule \( \Omega_D \) of \( \Omega_C \), hence \( e(D) = e(C) \) and \( \Sigma(D) = \Sigma(C) \), while by construction \( s(D) \) is at most \( \Sigma(C) \). Since we always have \( \Sigma(D) \leq s(D) \), we must have equality:

\[
s(D) = \Sigma(D) = \Sigma(C).
\]

This proves 2.

Now let \( D \) be as in 3. By 2, there exists a curve \( E \subseteq D \) such that \( e(E) = e(D) \) and \( s(E) = \Sigma(D) \). By hypothesis we have \( s(E) \geq t \), therefore

\[
\Sigma(C) \geq \Sigma(D) = s(E) \geq t
\]

and we are done.

**Theorem 2.8.2** Assume the ground field has characteristic zero. Let \( C \) be a curve in \( \mathbb{P}^3 \) with invariants \( e = e(C) \) and \( \Sigma = \Sigma(C) \). Let \( m \) be a positive integer satisfying \( m \leq \min\{\Sigma, (e + 4)/2\} \). Then for a general line \( L \), the principal spectrum \( sp_{C,L} \) contains the spectrum of the complete intersection of two surfaces of degrees \( m \) and \( e + 4 - m \).

**Remark 2.8.3** Somewhat more explicitly, the Theorem says that for \( 1 \leq t \leq \Sigma(C) \) we have

\[
h_C(n) \geq t + \mu_{C,L}(n) \quad \text{for } t - 1 \leq n \leq e + 3 - t
\]

where \( \mu_{C,L}(n) \) is the number of minimal generators of the Rao module \( M_C \) over the polynomial ring \( S_L \) in degree \( n \). This is equivalent to the statement of the Theorem because of the explicit calculation of the spectrum of a complete intersection of Corollary 1.5.3.

Note that \( \mu_{C,L}(n) \geq \mu_C(n) \), hence, by taking \( t = 1 \), we obtain a stronger version of 1 in Theorem 1.7.1. By taking \( t = 2 \), we see that if \( h_C(l) = 1 + \mu_{C,L}(l) \) for some \( l \) satisfying \( 1 \leq l \leq e + 1 \), then \( C \) contains a plane curve \( P \) with \( e(P) = e(C) \), and, as in the proof of Theorem 1.7.1, we conclude that \( h_C(n) = h_P(n) = 1 \) for \( l \leq n \leq e + 2 \); this strengthen statement 2 of Theorem 1.7.1.

For \( t \geq 3 \) we get a new statement which has no counterpart in Theorem 1.7.1. We have seen that a version of Theorem 1.7.1, namely Theorem 2.5.8, is valid for sheaves, so it is an interesting open problem whether Theorem 2.8.2 can be somehow generalized to the sheaf case; Theorem 2.7.3 may be thought as the case \( t = 0 \).

**Proof of Theorem 2.8.2**

Let \( m_0 \) be the largest integer \( m \) such that \( m \leq \min\{\Sigma, (e + 4)/2\} \). For \( 1 \leq m \leq m_0 \), the spectrum of the complete intersection \( X \) of two surfaces of
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degrees \( m_0 \) and \( e+4-m_0 \) contains the spectrum of the complete intersection of two surfaces of degree \( m \) and \( e+4-m \). Hence it is enough to prove the Theorem for \( m = m_0 \).

We now reduce the problem to the case \( s(C) = \Sigma(C) \). By Lemma 2.8.1, there is a curve \( D \subseteq C \) such that \( s(D) = \Sigma(D) = \Sigma(C) \). Now if \( L \) is a line skew to \( C \), then \( sp_{C,L} \) contains \( sp_{D,L} \) by Proposition 2.1.5. Hence it is enough to prove the Theorem for \( D \), that is, we may assume \( s(C) = \Sigma(C) \).

We have two cases to consider. First suppose that \( s(C) \leq e+4-s(C) \) so that \( m = s(C) \). Pick a surface \( Q \) of degree \( s(C) \) containing \( C \) and an element \( \xi \in H^0\omega_C(-e) \) which is not killed by any form of degree smaller than \( s(C) \): such a \( \xi \) exists because \( \Sigma(C) = s(C) \). Now we can use Proposition 2.2.3 to conclude that either \( C \) is obtained by an elementary biliaison of height \( h = e+4-s(C) \) on \( Q \) from some curve \( C_0 \), or \( C \) is a complete intersection of two surfaces of degrees \( s(C) \) and \( e+4-s(C) \); then by Proposition 2.2.1 the principal spectrum of \( C \) relative to a general line contains the spectrum of the complete intersection of two surfaces of degree \( s(C) \) and \( e+4-s(C) \) (note that so far we have not used the characteristic zero hypothesis).

The second case we have to consider is when \( s(C) > e+4-s(C) \). In this case, \( m \) is equal to the integral part of \((e+4)/2\). We write

\[
2m = e+4+c_1
\]

with \( c_1 = 0 \) or \(-1\). We pick \( \xi \in H^0\omega_C(-e) \) as above so that \( \Sigma(\xi) = \Sigma(C) \). Then \( \xi \) defines an extension

\[
0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(c_1-m) \longrightarrow \mathcal{F} \longrightarrow \mathcal{J}_C(m) \longrightarrow 0.
\]

We have \( m - c_1 \geq m \geq 1 \) because \( e \geq -2 \), while

\[
(m - c_1) + c_1 = m \leq \Sigma - 1 = \Sigma(\xi) - 1
\]

by construction, so that the map

\[
H^0\mathcal{O}_{\mathbb{P}^3}(m) \rightarrow H^0\omega_C(4+c_1-m)
\]

which sends \( g \) to \( g \cdot \xi \) is injective. It follows from Corollary 2.6.3 that \( \mathcal{F} \) is a curvy sheaf (here we are using the characteristic zero hypothesis: we need the Grauert-M"{u}lich Theorem to ensure that stable implies curvy). Now Proposition 2.7.11 tells us that for a general line \( L \) the principal spectrum \( sp_{C,L} \) contains the spectrum of the complete intersection of two surfaces of degrees \( m \) and \( m-c_1 = e+4-m \). This concludes the proof.
For an integral curve \( C \), the statement of Theorem 2.8.2 remains true in any characteristic, if we replace the principal spectrum with the full spectrum, as we can use Hartshorne’s characteristic free theory of the spectrum for rank 2 reflexive sheaves.

**Proposition 2.8.4** Let \( C \) be an integral curve in \( \mathbb{P}^3 \) with invariants \( e = e(C) \) and \( s = s(C) \). If \( m \) is a positive integer satisfying \( m \leq \min\{s, (e + 4)/2\} \), the spectrum of \( C \) contains the spectrum of the complete intersection of two surfaces of degrees \( m \) and \( e + 4 - m \).

**Proof** The proof is perfectly analogous to that of Theorem 2.8.2. Since \( C \) is integral, the sheaf \( \mathcal{F} \) in the proof is in this case a rank two reflexive sheaf with \( c_1 = 0 \) or \(-1\) satisfying \( H^0(\mathcal{F}) = 0 \), and we can use Proposition 2.6.4 to obtain the desired conclusion.

As a consequence of Theorem 2.8.2, we obtain a version of the Speciality Theorem of Halphen-Gruson-Peskine for Cohen-Macaulay curves:

**Theorem 2.8.5** Assume the ground field has characteristic zero. Let \( C \) be a curve in \( \mathbb{P}^3 \) with invariants \( e = e(C) \) and \( \Sigma = \Sigma(C) \). Then for all positive integers \( m \) satisfying \( m \leq \min\{\Sigma, (e + 4)/2\} \) we have:

1. \( \deg C \geq m(e + 4 - m) \) with equality holding if and only if \( C \) is a deformation with constant cohomology of a complete intersection of two surfaces of degrees \( m \) and \( e + 4 - m \).

2. \( s(C) \leq \deg C - m(e + 3 - m) \)

**Proof** If \( m \leq \min\{\Sigma, (e+4)/2\} \), Theorem 2.8.2 tells us that the principal spectrum of \( C \) contains the spectrum of the complete intersection \( X \) of two surfaces of degrees \( m \) and \( e + 4 - m \); in particular, \( \deg C \geq \deg X = m(e + 4 - m) \).

Suppose equality holds: then in the chain of inclusions

\[ sp_X \subseteq sp_{C,L} \subseteq sp_C \]

the two finite sets on the left and on the right have the same number of elements, hence we have equality everywhere. Equality on the right tells us that \( C \) is ACM. For an ACM curve, the spectrum is equivalent to the postulation, and the set of ACM curves with given postulation is irreducible. Statement 1 now follows.
To prove 2, we fix a general line \( L \) of equations \( x = y = 0 \) and a linear form \( z \) mapping to a general linear form of \( R/(x, y) \). The proof of Proposition 1.8.1 shows that \( s(C) \) is less or equal than the number of generators of \( A_C \) over \( k[x, y, z] \). Thus it is enough to show that the number of generators of \( A_C \) over \( k[x, y, z] \) is at most \( \deg C - m(e + 3 - m) \). To this end, we look at the exact sequence defining the principal spectrum:

\[
0 \rightarrow P(C, L) \rightarrow A_C \otimes_S k \rightarrow M_C \otimes_S k \rightarrow 0
\]

where \( S = S_L = k[x, y] \), and we let \( \phi : P(C, L) \rightarrow P(C, L) \) denote the multiplication by the linear form \( z \). The number of generators of \( A_C \) over \( S \) is equal to \( \dim A_C \otimes_S k = \deg C \), hence the number of generators of \( A_C \) over \( k[x, y, z] \) is at most \( \deg C \) minus the dimension of the image of \( \phi \). Now by Proposition 1.6.11 applied with \( V = P(C, L) \), we know that \( \phi_n : P(C, L)_n \rightarrow P(C, L)_{n+1} \) has maximal rank for all \( n \), so that

\[
\dim \text{Im} (\phi_n) = \min\{\dim P_n, \dim P_{n+1}\}.
\]

By Theorem 2.8.2 we have \( \dim P_n \geq h_X(n) \), where \( X \) is the complete intersection of two surfaces of degree \( m \) and \( e + 4 - m \). Thus

\[
\dim \text{Im} (\phi_n) \geq \min\{h_X(n), h_X(n+1)\} = h_Y(n)
\]

where \( Y \) is the complete intersection of two surfaces of degrees \( m \) and \( e + 3 - m \). Summing over \( n \) we find

\[
\dim \text{Im} (\phi) \geq \deg Y = m(e + 3 - m),
\]

hence the number of generators of \( A_C \) over \( k[x, y, z] \) is at most \( \deg C - m(e + 3 - m) \), and we are done.

As a Corollary we obtain the Speciality Theorem of Halphen-Gruson-Peskine for integral curves in \( \mathbb{P}^3 \). Gruson and Peskine proved the result more generally for codimension two subschemes of arbitrary dimension: see [15], Theorem 1.2.

**Corollary 2.8.6 (Halphen-Gruson-Peskine)** Assume the ground field has characteristic zero. Let \( C \) be an integral curve in \( \mathbb{P}^3 \) with index of speciality \( e = e(C) \). Then

\[
s(C) \leq \max\{0, \deg C - n(e + 4 - n)\} + n
\]

for all positive integers \( n \).
CHAPTER 2. THE SPECTRUM OF A CURVE IN \( \mathbb{P}^3 \)

Proof Since \( C \) is integral, we have \( \Sigma(C) = s(C) \). We have several possibilities to consider: if \( n \leq \min\{s(C), (e+4)/2\} \), by Theorem 2.8.5 we have \( \deg C \geq n (e+4-n) \) and

\[
s(C) \leq \deg(C) - n (e+3-n) = \max\{0, \deg C - n (e+4-n)\} + n.
\]

If \( n \geq s(C) \), the statement is obvious. If \( (e+4)/2 < n < s(C) \), we have two cases: in the first case we have \( n \leq e+4 \) and we can set \( m = e+4 - n \) and apply Theorem 2.8.5 as above, keeping in mind that \( m < n \) by construction. In the second case we have \( n \geq e+4 \), so that the right hand side \( \max\{0, \deg C - n (e+4-n)\} + n \) is at least \( \deg C \) which is always an upper bound for \( s(C) \).

Remark 2.8.7 Note that Proposition 2.8.4 is not strong enough to prove Corollary 2.8.6; we need the full strength of Theorem 2.8.2.

We have shown in Proposition 1.8.1 that for a curve \( C \subseteq \mathbb{P}^3 \) we always have \( s(C) \leq \deg(C) - e(C) - 2 \), and for all positive integers \( s \) and \( d \) satisfying \( s \leq d \) we have constructed a curve \( C \) such that \( \deg(C) = d \), \( s(C) = s \), \( e(C) = d - s - 2 \) and containing a plane curve of degree \( d - s + 1 \). We can now show that all curves \( C \) satisfying \( s(C) = \deg(C) - e(C) - 2 \) and \( e(C) \geq 1 \) contain a plane curve of degree \( d - s + 1 \). To describe these curves a little more precisely, we need the definition (taken from [27]) of the residual scheme to a given subscheme of \( \mathbb{P}^N \) with respect to a hypersurface.

Definition 2.8.8 Let \( F \) be a hypersurface in \( \mathbb{P}^N \) defined by the equation \( f = 0 \) and let \( X \) be a closed subscheme of \( \mathbb{P}^N \). We define the residual scheme \( Y = \text{Res}_F(X) \) to \( X \) with respect to \( F \) to be the subscheme corresponding to the ideal sheaf

\[
\mathcal{J}_Y = f^{-1} \text{Ker} (\mathcal{J}_{X, \mathbb{P}^N} \longrightarrow \mathcal{J}_{X \cap F, F})
\]

so that we have an exact sequence

\[
0 \longrightarrow \mathcal{O}_Y(-d) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{X \cap F} \longrightarrow 0
\]

where \( d = \deg F \).

We will be interested in the case \( X \) is a curve \( C \) in \( \mathbb{P}^3 \) and \( F \) is a plane \( H \) of equation \( h = 0 \). Then \( Y = \text{Res}_H(C) \) is a curve contained in \( C \): the associated points of \( Y \) are among the associated points of \( C \), as one sees from the inclusion \( \mathcal{O}_Y(-1) \hookrightarrow \mathcal{O}_C \), hence \( Y \) doesn’t have zero dimensional associated points.
Proposition 2.8.9 Assume the ground field has characteristic zero. Let \( C \) be a curve in \( \mathbb{P}^3 \), and let \( d = \deg(C), s = s(C) \) and \( e = e(C) \). Suppose \( s = d - e - 2 \) and \( e \geq 1 \). Then

1. \( C \) contains a plane curve \( P \) of degree \( d - s + 1 \) (so that \( e(P) = e(C) \));
2. if \( H \) is the plane containing \( P \) and \( Y = \text{Res}_H(C) \) is the residual curve, we have \( s(Y) = \deg(Y) = s - 1 \);
3. the spectrum of \( C \) satisfies
   \[
   h_C(n) = 1 \quad \text{for } 2 \leq n \leq e + 2.
   \]

Proof We first show that \( \Sigma(C) \) must be one: otherwise, we could take \( m = 2 \) in Theorem 2.8.5 and conclude that
\[
s \leq d - 2e - 2 < d - e - 2,
\]
a contradiction. Hence \( \Sigma_C = 1 \) and by Lemma 2.8.1 \( C \) contains a plane curve \( P \) with \( e(P) = e \); hence \( \deg(P) = e(P) + 3 = d - s + 1 \). Since \( \deg(P) \geq 4 \), there is a unique plane \( H \) containing \( P \). Furthermore, since \( e(P) = e(C) \), \( P \) is the largest curve contained in \( C \cap H \). Let \( Y = \text{Res}_H(C) \). From the exact sequence:
\[
0 \rightarrow J_Y(-1) \rightarrow J_C \rightarrow J_{C \cap H, H} \rightarrow 0
\]
we see that \( \deg(Y) = \deg(C) - \deg(P) = s - 1 \) and that \( s(Y) \geq s - 1 \); since we always have \( s(Y) \leq \deg(Y) \), we conclude that \( s(Y) = \deg(Y) = s - 1 \). Next we look at the exact sequence
\[
0 \rightarrow \mathcal{O}_Y(-1) \rightarrow \mathcal{O}_C \rightarrow \mathcal{O}_{C \cap H} \rightarrow 0
\]
from which we obtain
\[
0 \rightarrow \Omega_P \rightarrow \Omega_C \rightarrow \Omega_Y(1)
\]
since \( P \) is equal to \( C \cap H \) minus its zero dimensional associated points. As \( Y \) satisfies \( s(Y) = \deg(Y) \), we have \( e(Y) = -2 \), therefore \( \Omega_P \) and \( \Omega_C \) are equal in degrees \( \leq 0 \). By Proposition 1.4.9 this implies
\[
h_C(n) = h_P(n) = 1
\]
for \( 2 \leq n \leq e + 2 \), and we are done.

We can make a few remarks on curves \( Y \) satisfying \( s(Y) = \deg(Y) \). By Proposition 1.8.1 we know that such a curve has index of speciality
$e(Y) = -2$; hence by (1.7.6) the support of $Y$ is a disjoint union of lines, and on each line in its support $Y$ is a quasiprimitive multiplicity structure which does not contain a planar double structure. In fact, the support of $Y$ is either a line or two disjoint lines. To prove this we first show:

**Lemma 2.8.10** Let $C$ be a curve in $\mathbb{P}^3$, and let $C_1, C_2, \ldots, C_r$ be the irreducible components of $C$ with the induced scheme structure. Let $D = C_{\text{red}}$ and $D_j = (C_j)_{\text{red}}$, and denote by

$$\mu_j = \deg(C_j)/\deg(D_j)$$

the multiplicity of $C$ along $D_j$. Then

$$s(C) \leq s(D) + \sum_{j=1}^r (\mu_j - 1) s(D_j)$$

**Proof** Let $f$ be the equation of a surface containing $D$, and let $f_j$ be the equation of a surface containing $D_j$, for $j = 1, \ldots, r$. Then

$$g = f \times f_1^{\mu_1-1} \times \cdots \times f_r^{\mu_r-1}$$

is the equation of a surface containing $C$.

**Proposition 2.8.11** If $Y \subseteq \mathbb{P}^3$ is a curve satisfying $s(Y) = \deg(Y)$, the support of $Y$ is either a line or two disjoint lines. If $X$ is a connected component of $Y$, then $s(X) = \deg(X)$ and $X$ is a quasiprimitive multiplicity structure on a line which does not contain a planar double structure.

**Proof** If $s(Y) = \deg(Y)$, it follows from Lemma 2.8.10 that $s(Y_{\text{red}}) = \deg(Y_{\text{red}})$ and $s(X) = \deg(X)$ for all irreducible components $X$ of $Y$, as for any curve $D$ we have $s(D) \leq \deg(D)$. On the other hand, we know that $Y_{\text{red}}$ is a disjoint union of $t$ lines. Now a disjoint union of $t$ lines has spectrum $\{0\}$, which gives

$$h^0Y(t-1) < h^0\mathcal{O}_{\mathbb{P}^3}(t-1) \quad \text{for} \quad t \geq 3.$$  

Hence $s(Y_{\text{red}}) = t$ forces $t$ to be either 1 or 2, and we are done.

The classification of curves $C$ satisfying $s(C) = \deg(C) - e(C) - 2$ is more difficult when $e(C) \leq 0$ because it involves a number of special cases. However, for integral curves it is easy: using Theorem 2.8.6 and Theorem 1.8.3 we see that an integral curve $C$ satisfies $s(C) = \deg(C) - e(C) - 2$ if and only if it is one of the following: a plane curve, a twisted cubic curve, a rational quartic curve, a rational quintic curve of maximal rank, an integral curve of arithmetic genus 1 and degree 4 or 5.
Chapter 3

The Spectrum of an Integral Curve

3.1 The Spectrum of an Integral ACM Curve in $\mathbb{P}^3$.

One of the most interesting problems in the theory of the spectrum is that of determining exactly which sets of integers occur as spectra of integral curves (or, equivalently by Bertini’s Theorem, as spectra of integral quasi ACM subschemes). No complete solution to this problem is known, but there are some results in the literature (modulo the language) which we are going to review, and sometimes generalize, in the first few sections of this chapter. We will then use these results and those of the previous chapter to determine all possible spectra of integral curves of degree at most 11 in $\mathbb{P}^N$ (for any $N \geq 1$).

We start with a theorem of Gruson and Peskine which describes the spectra of integral ACM curves in $\mathbb{P}^3$. More precisely, Gruson and Peskine [15] compute the possible numerical characters of integral ACM curves in $\mathbb{P}^3$, and show that they occur as characters of smooth irreducible ACM curves in characteristic zero; Nollet [39] shows that their result holds in characteristic $p$ as well. An analogous result is in [31]. For an ACM curve $C$ in $\mathbb{P}^3$, the numerical character and the spectrum of $C$ are equivalent invariants, and in section 1.6 we have seen how to compute one out of the other. We can therefore translate the result of Gruson and Peskine into the language of the spectrum.

Recall that, given an integer $s \geq 1$, we have defined an $s$-sequence to be
a sequence of integers \( h(n) \) satisfying the following properties:

\[
  h(n) = \begin{cases} 
    n + 1 & \text{if } 0 \leq n \leq s - 1 \\
    h(n + 1) & \text{if } s - 1 \leq n \\
    0 & \text{if } n \gg 0 \text{ or if } n < 0.
  \end{cases}
\]

Given an \( s \)-sequence \( h(n) \), we define \( t = t(h) \geq s \) to be the least integer \( n \geq s \) such that \( h(t) < s = h(s - 1) \).

**Definition 3.1.1** ([31]) We say that an \( s \)-sequence \( h(n) \) is of decreasing type if it decreases strictly until it reaches zero for \( n \geq t(h) \).

**Example 3.1.2** \( \{1, 2, 3, 3, 3, 1\} \) (meaning \( h(0) = 1, h(1) = 2, \text{etc.} \)) is a 3-sequence of decreasing type with \( t = 5 \), while \( \{1, 2, 3, 2, 2\} \) is a 3-sequence not of decreasing type, with \( t = 3 \).

**Theorem 3.1.3** ([15, 31, 39]) If \( C \) is an integral ACM curve in \( \mathbb{P}^3 \) with spectrum \( \{n^{h_C(n)}\} \), then \( h_C(n) \) is an \( s(C) \)-sequence of decreasing type. Furthermore, if \( h(n) \) is an \( s \)-sequence of decreasing type, then \( \{n^{h(n)}\} \) is the spectrum of some smooth irreducible ACM curve in \( \mathbb{P}^3 \) with \( s(C) = s \).

### 3.2 On a Theorem of Stanley

The main result in this section, Theorem 3.2.2 below, is a variant of a Theorem of Stanley ([48], Theorem 2.1) which gives some information on the \( h \)-vector of a class of graded Cohen-Macaulay domains. We state and prove Stanley’s Theorem only in the case of curves, but without assuming the curves to be integral. A version of Stanley’s Theorem is in [8], Theorem 4.3.11 on page 172. The idea of the proof of all these results is already in [21], Lemma 7.4 p.153.

We first establish the following Lemma:

**Lemma 3.2.1** Let \( C \subseteq \mathbb{P}^N \) be a curve. Let \( \xi \in H^0\omega_C(n) \), and let \( \phi : \mathcal{O}_C \to \omega_C(n) \) the morphism defined by \( \xi \). The following conditions are equivalent:

1. the morphism \( \phi \) is injective;

2. the global section \( \xi \) generates the sheaf \( \omega_C(n) \) except at finitely many closed points.

Furthermore, if \( N = 3 \), the above conditions are equivalent to the following: the sheaf \( \mathcal{F} \) corresponding to the pair \( (C, \xi) \) as in Proposition 2.6.1 is reflexive.
3.2. ON A THEOREM OF STANLEY

Proof  Note that \( \xi \) generates the sheaf \( \omega_C(n) \) except at finitely many points if and only if \( \phi \) is surjective at the generic points of \( C \).

Since \( C \) is locally Cohen-Macaulay, if \( \phi \) is not injective the associated points of \( \text{Ker} \phi \) have dimension one; hence \( \phi \) is injective if and only if it is injective at the generic points of \( C \). Now, if \( \eta \) is a generic point of \( C \), we have an isomorphism 
\[
\omega_{C, \eta} \cong \text{Hom}_k (\mathcal{O}_{C, \eta}, k)
\]
because the canonical module localizes and \( \mathcal{O}_{C, \eta} \) is an artinian \( k \)-algebra. In particular, \( \omega_{C, \eta} \) and \( \mathcal{O}_{C, \eta} \) have the same dimension over \( k \), so that \( \phi \) is injective if and only if it is surjective. We conclude that \( \phi \) is injective if and only if it is generically surjective, thus 1 and 2 are equivalent.

The equivalence of 2 and 3 when \( N = 3 \) is due to Hartshorne ([21],Theorem 4.1).

Theorem 3.2.2  Let \( C \) be a curve in \( \mathbb{P}^N \). Suppose there is a global section \( \xi \in H^0 \omega_C(-t) \) which generates the sheaf \( \omega_C(-t) \) except at finitely many points. Then for all integers \( n \) we have:
\[
\sum_{k \leq n} h_C(k) \leq \sum_{k \leq n} h_C(2 + t - k)
\]

Proof  By Lemma 3.2.1 \( \xi \) defines an injective morphism \( \phi : \mathcal{O}_C \rightarrow \omega_C(-t) \), and taking global sections we obtain an exact sequence:
\[
0 \rightarrow A_C \rightarrow \Omega_C(-t) \rightarrow Q \rightarrow 0.
\]
\( A_C \) and \( \Omega_C \) are Cohen-Macaulay modules of dimension two over the coordinate ring \( R \) of \( \mathbb{P}^N \); in particular, they have depth two over the irrelevant maximal ideal, and therefore \( Q \) has depth at least one if it is nonzero. On the other hand, as \( \phi \) is an isomorphism at the generic points of \( C \), \( Q \) has dimension at most 1. Hence \( Q \) is either zero or a Cohen-Macaulay \( R \)-module of dimension 1, so that we can choose a linear form \( x \in R_1 \) which is \( Q \)-regular. Then \( Q \) is free of finite rank as a \( k[x] \)-module, and for all integers \( n \) we have
\[
\partial \dim Q_n = \dim Q_n - \dim Q_{n-1} \geq 0,
\]
that is,
\[
(3.1) \quad \partial h^0 \omega_C(n - t) - \partial h^0 \mathcal{O}_C(n) \geq 0.
\]
Now recall that \( \partial^2 h^0 \mathcal{O}_C(n) = h_C(n) \), while \( \partial^2 h^0 \omega_C(n - t) = h_C(2 + t - n) \) by Proposition 1.4.9. Therefore
\[
\partial h^0 \mathcal{O}_C(n) = \sum_{k \leq n} h_C(k)
\]
and
\[ \partial h^0 \omega_C(n-t) = \sum_{k \leq n} h_C(2 + t - k). \]

Thus our claim follows from (3.1).

**Remark 3.2.3** The largest we can take \( t \) in the Theorem 3.2.2, the strongest result we obtain. The best case is when we can take \( t = e(C) \), which always happens if \( C \) is integral.

The following Corollary, in case \( C \) is an integral curve, is a special case of [48], Theorem 2.1.

**Corollary 3.2.4 (Stanley)** Let \( C \) be a curve in \( \mathbb{P}^N \) with index of speciality \( e \). Suppose that there is a global section \( \xi \in H^0 \omega_C(-e) \) which generates the sheaf \( \omega_C(-e) \) except at finitely many points (this is always the case if \( C \) is integral). Then \( h_C(k) = 0 \) for \( k < 0 \), and for all integers \( n \geq 0 \) we have

\[ \sum_{k=0}^{n} h_C(k) \leq \sum_{k=0}^{n} h_C(e + 2 - k). \]

**Proof** By Theorem 3.2.2 we have

\[ \sum_{k \leq n} h_C(k) \leq \sum_{k \leq n} h_C(2 + e - k) \]

for all integers \( n \); taking \( n = -1 \) and keeping in mind that \( h_C(l) = 0 \) for \( l > e+2 \), we see that \( h_C(k) = 0 \) for \( k < 0 \) (this can also be seen as follows: if a homogeneous morphism from \( A_C \) to \( \Omega_C \) is injective and maps the identity to an element of minimal degree in \( \Omega_C \), then \( A_C \) cannot have elements of negative degrees, which is the same as saying that \( h_C(k) = 0 \) for \( k < 0 \)). Since \( h_C(k) = 0 \) for \( k < 0 \) and \( k > e+2 \), we can rewrite (3.2) in the form

\[ \sum_{k=0}^{n} h_C(k) \leq \sum_{k=0}^{n} h_C(e(C) + 2 - k) \]

and we are done.

**Example 3.2.5** If we take \( n = 0 \), we obtain \( h_C(0) \leq h_C(e+2) \); if we take \( n = 1 \), we obtain \( h_C(0) + h_C(1) \leq h_C(e+2) + h_C(e+1) \), and so on.

We now show that the conclusion of Corollary 3.2.4 implies Clifford’s Theorem for the sheaves \( \mathcal{O}_C(l) \) if \( h^0 \mathcal{O}_C = 1 \).
Proposition 3.2.6 Let $C \subseteq \mathbb{P}^N$ be a curve, not necessarily integral, with index of speciality $e$. Suppose $h^0\mathcal{O}_C = 1$ and
\[ \sum_{k=0}^{n} h_C(k) \leq \sum_{k=0}^{n} h_C(e + 2 - k) \]
for all integers $n \geq 0$. Then
\[ h^0\mathcal{O}_C(l) - 1 \leq \frac{\deg \mathcal{O}_C(l)}{2} \quad \text{if} \quad 0 \leq l \leq e. \]

In particular, $e \deg(C) \leq 2g(C) - 2$.

Proof Note that the condition $h^0\mathcal{O}_C = 1$ means $h_C(0) = 1$ and $h_C(n) = 0$ for $n < 0$, and this ensures $g = h^0\omega_C \geq 0$. For an invertible sheaf $\mathcal{L}$ on $C$, we define
\[ \text{Cliff} \mathcal{L} = \deg \mathcal{L} - 2h^0\mathcal{L} + 2. \]
We need to show that $\text{Cliff} \mathcal{O}_C(l) \geq 0$ for $0 \leq l \leq e$. Now
\[ \text{Cliff} \mathcal{O}_C(l) - \text{Cliff} \mathcal{O}_C(l - 1) = \deg C - 2h^0\mathcal{O}_C(l) = \deg C - 2 \sum_{k=0}^{l} h_C(k) \]
is a decreasing function of $l$, hence
\[ \text{Cliff} \mathcal{O}_C(l) \geq \min(\text{Cliff} \mathcal{O}_C, \text{Cliff} \mathcal{O}_C(e)) \]
for $0 \leq l \leq e$. As $\text{Cliff} \mathcal{O}_C = 0$, we are reduced to show $\text{Cliff} \mathcal{O}_C(e) \geq 0$, and, as in the proof of Clifford’s Theorem ([20], Theorem 5.4 page343), we just need to prove that
\[ h^0\mathcal{O}_C(e) + h^1\mathcal{O}_C(e) \leq g + 1. \]

Now we have the formulas
\[ g = h^0\omega_C = \sum_{k=0}^{e} (e + 1 - k)h_C(e + 2 - k) \]
\[ h^0\mathcal{O}_C(e) = \sum_{k=0}^{e} (e + 1 - k)h_C(k) \]
\[ h^1\mathcal{O}_C(e) = h_C(e + 2) \]
hence
\[ g + 1 - h^0 \mathcal{O}_C(e) - h^1 \mathcal{O}_C(e) = \sum_{k=0}^n (e + 1)(h_{C}(e + 2 - k) - h_{C}(k)) + h_{C}(0) - h_{C}(e + 2) \]

This is a sum with positive weights (either 1 or 2) of terms of the form
\[ \sum_{k=0}^n (h_{C}(e + 2 - k) - h_{C}(k)) \]
which are nonnegative by assumption. Thus \( g + 1 - h^0 \mathcal{O}_C(e) - h^1 \mathcal{O}_C(e) \geq 0 \), which proves our first statement.

To show \( e \deg(C) \leq 2g - 2 \), we look at the sheaf \( \mathcal{O}_C(e) \). By definition of \( e \) we have \( h^1 \mathcal{O}_C(e) \geq 1 \); if \( h^0 \mathcal{O}_C(e) = 0 \), then by Riemann-Roch
\[ e \deg(C) = \deg \mathcal{O}_C(e) = -h^1 \mathcal{O}_C(e) - 1 + g \leq g - 2 \leq 2g - 2. \]
If \( h^0 \mathcal{O}_C(e) \geq 1 \), then \( e \geq 0 \) and by what we have just proven
\[ h^0 \mathcal{O}_C(e) - 1 \leq \frac{\deg \mathcal{O}_C(e)}{2}, \]
while by Riemann-Roch
\[ h^0 \mathcal{O}_C(e) - 1 = h^1 \mathcal{O}_C(e) + \deg \mathcal{O}_C(e) + 1 - g - 1 \geq 1 - g + \deg \mathcal{O}_C(e). \]
Hence \( g - 1 \geq \deg \mathcal{O}_C(e)/2 = (e \deg(C))/2 \) and we are done.

Remark 3.2.7 Proposition 3.2.6 is useful in the following context: suppose we are given a set of integers with multiplicities \( sp \), and we are asked if there is an integral curve \( C \) with spectrum \( sp \). If the answer is yes, then \( sp \) must be such that the hypotheses of the Proposition are satisfied; on the other hand, if we see that these hypotheses are satisfied, then we don’t need to check if Clifford’s Theorem or the inequality \( e \deg(C) \leq 2g(C) - 2 \) hold.

Remark 3.2.8 On the other hand, even if \( C \) is a curve for which \( \text{Cliff} \mathcal{O}_C(e) \geq 0 \) and \( h^0 \mathcal{O}_C = 1 \), the spectrum of \( C \) may fail to satisfy Stanley’s inequality
\[ \sum_{k=0}^n h_C(k) \leq \sum_{k=0}^n h_C(e + 2 - k). \]
Take for example a curve \( C \) with spectrum
\[
\text{sp}_C = \{0, 1^2, 2^6, 3^3, 4^3, 5^2\}
\]
If such a curve exists (which we haven’t checked), it has \( e = 3 \) and \( \text{Cliff}_C(3) = 51 - 50 + 2 = 3 \), but it doesn’t satisfy Stanley’s condition as
\[
1 + 2 + 6 > 3 + 3 + 2.
\]
In particular, such a curve, if it exists, cannot be integral.

In case \( N = 3 \), as a Corollary of Theorem 3.2.2 we obtain a condition the spectrum of semistable a rank 2 reflexive sheaf on \( \mathbb{P}^3 \) must satisfy. For a rank two vector bundle this condition is a trivial consequence of the symmetry of the spectrum ([21], Proposition 7.2), but for general reflexive sheaves it is interesting, and apparently it has not been noticed before, even though it would follow from the proof of Lemma 7.4 in [21]. While the problem of determining the possible spectra of rank two semistable reflexive sheaves is wide open, the following Corollary contributes a necessary condition such spectra must satisfy.

**Corollary 3.2.9** Let \( \mathcal{F} \) be a reflexive sheaf of rank 2 on \( \mathbb{P}^3 \) normalized so that \( c_1 = c_1(\mathcal{F}) \) is either 0 or \(-1\). Suppose that \( H^0(\mathbb{P}^3, \mathcal{F}(-1)) = 0 \), and let \( \{n^{h_F(n)}\} \) be the spectrum of \( \mathcal{F} \). Then for all integers \( n \) we have
\[
\sum_{k \leq n} h_F(k) \leq \sum_{k \geq -n} h_F(k - 2 - c_1).
\]

**Proof** The hypothesis \( H^0(\mathbb{P}^3, \mathcal{F}(-1)) = 0 \) ensures that the spectrum of \( \mathcal{F} \) is well defined. We can pick an integer \( p \) so that there is a global section \( s \in H^0(\mathbb{P}^3, \mathcal{F}(p)) \) whose zero set is a Cohen-Macaulay curve \( C \)—for example we can take \( p \) to be the smallest integer \( n \) such that \( H^0(\mathbb{P}^3, \mathcal{F}(n)) \neq 0 \). By Proposition 2.6.4, for all integers \( k \) we have
\[
(3.3) \quad h_C(k) = h_F(k - c_1 - p) + h_X(k)
\]
where \( X \) is the complete intersection of two surfaces of degrees \( p + c_1 \) and \( p \) (with the convention \( h_X(k) = 0 \) if \( p + c_1 \leq 0 \)). By Proposition 2.6.1 the pair \( (\mathcal{F}, s) \) defines a global section \( \xi \in H^0(\omega_C(-e(X))) \), where \( e(X) = 2p + c_1 - 4 \). Since \( \mathcal{F} \) is reflexive, Lemma 3.2.1 tells us that \( \xi \) generates \( \omega_C(-e(X)) \) except at finitely many points, hence, by Theorem 3.2.2, for all integers \( n \) we have
\[
\sum_{k \leq n} h_C(k) \leq \sum_{k \leq n} h_C(e(X) + 2 - k).
\]
We rewrite this inequality using (3.3); since $X$ is subcanonical of index $e(X)$, its spectrum satisfies $h_X(k) = h_X(e(X) + 2 - k)$ by Proposition 1.4.14, so we can cancel the contribution of $h_X$ and conclude that

$$\sum_{k \leq n} h_X(k - c_1 - p) \leq \sum_{k \leq n} h_X(2 + e(X) - k - c_1 - p)$$

for all integers $n$. Since $2 + e(X) - 2c_1 - 2p = -2 - c_1$, this proves the Corollary.

### 3.3 The Uniform Position Theorem and Another Result of Stanley

The main result of this section is Theorem 3.3.5, which was proven by Stanley [48] for integral ACM subschemes of $\mathbb{P}^N$ over a field of characteristic zero. We show that his proof works equally well for integral quasi ACM subschemes. Theorem 3.3.5 improves the results of the previous section for integral curves. In [13] a weaker but similar result is proven in arbitrary characteristic.

Stanley’s proof is based on the Uniform Position Theorem of Castelnuovo-Harris, so we begin this section by reviewing the basic definitions about linear systems on an integral curve, and by stating the Uniform Position Theorem. The Uniform Position Theorem holds for all integral curves when the ground field has characteristic zero, while Rathmann [45] has shown that in characteristic $p$ it is still valid in many important cases, e.g., for all smooth irreducible nondegenerate curves in $\mathbb{P}^N$ if $N \geq 4$. The results of this section hold for all integral curves for which the Uniform Position Theorem is valid.

Let $C$ be an integral projective curve. If $\mathcal{L}$ is an invertible sheaf on $C$, there is a bijection between the set of effective Cartier divisors $D$ on $C$ such that $\mathcal{O}_C(D) \cong \mathcal{L}$ and the projective space of lines in $H^0(C, \mathcal{L})$, as any such $D$ arises as the divisor of zeros of a nonzero global section of $\mathcal{L}$. A complete linear system is the set of all effective Cartier divisors linearly equivalent to a given Cartier divisor $D$ and is denoted by the symbols $|D|$; it may be empty. As we have seen, a complete linear system has a structure of projective space, and we define a sublinear system $\mathcal{D}$ to be a linear subspace of a complete linear system $|D|$; thus $\mathcal{D}$ corresponds to a sub-vector space $V$ of $H^0(C, \mathcal{O}_C(D))$. The dimension of $\mathcal{D}$ is by the definition its dimension as a projective linear space, hence

$$\dim \mathcal{D} = \dim_k V - 1.$$
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The degree of $D$ is the degree of any of the divisors in $D$, provided that $D$ is nonempty.

Given two Cartier divisors $D$ and $E$, we write $D \geq E$ if $D - E$ is effective.

**Proposition 3.3.1** Let $P_1, \ldots, P_r$ be distinct smooth points of $C$ and consider the Cartier divisor $\Gamma = P_1 + \cdots + P_r$: given a linear system $D$ on $C$, we define

$$D - \Gamma = \{ D \in D : D \geq \Gamma \}$$

Then $D - \Gamma$ is a linear system with

$$\dim(D - \Gamma) \geq \dim D - r.$$ 

Furthermore, if $D \in D$, there is an injective linear map $D - \Gamma \to |D - \Gamma|$ defined by $E \mapsto E - \Gamma$; this is an isomorphism if $D$ is a complete linear system.

Given a linear system $D$ and a divisor $\Gamma = P_1 + \cdots + P_r$, where the $P_i$’s are distinct smooth points of $C$, we define $c(\Gamma, D)$ to be the number of conditions imposed by $\Gamma$ on $D$, that is

$$c(\Gamma, D) = \dim D - \dim D - \Gamma$$

Note that $c(\Gamma, D) \leq \deg \Gamma$; if equality holds we say that $\Gamma$ imposes independent conditions on the linear system $D$; furthermore, we can always find a sub-divisor $\Gamma' \subseteq \Gamma$ of degree $c(\Gamma, D)$ which imposes independent conditions on $D$:

$$c(\Gamma, D) = c(\Gamma', D) = \deg \Gamma'.$$

If $D'$ is a sub-linear system of $D$, then

$$c(\Gamma, D') \leq c(\Gamma, D).$$

We can now state the Uniform Position Theorem of Castelnuovo-Harris. For a proof the reader is referred to [1], Chapter 3.

**Theorem 3.3.2 (Uniform Position Theorem)** Assume the ground field $k$ has characteristic zero, and let $C \subseteq \mathbb{P}^N$, $N \geq 3$, be an integral curve which is nondegenerate (i.e., not contained in a hyperplane). Fix a linear system $D$ on $C$. There exists a dense Zariski open set of hyperplanes $U$ such that for all $H \in U$ the hyperplane section $\Gamma = C \cap H$ consists of smooth distinct points of $C$, and for all subsets $\Gamma'$ of $\Gamma$ we have

$$c(\Gamma', D) = \min (\deg \Gamma', c(\Gamma, D))$$

(or, equivalently, if $\Gamma'$ fails to impose independent conditions on $D$, then every divisor in $D$ containing $\Gamma'$ contains $\Gamma$ as well).
Given two linear systems \(D\) and \(E\), we define \(D + E\) to be the smallest linear system containing all divisors of the form \(D + E\) with \(D \in D\) and \(E \in E\); if \(D\) corresponds to the vector space \(V \subseteq H^0(C, O_C(D))\) and \(E\) to \(W \subseteq H^0(C, O_C(E))\), then \(D + E\) corresponds to the image of \(V \otimes W\) in \(H^0(C, O_C(D + E))\).

The following Corollary of the Uniform Position Theorem is what the proof of Corollary 3.5 in [18] proves. Its statement should be compared with that of the Bilinear Mapping Lemma ([21], Lemma 5.1).

**Corollary 3.3.3** Assume the ground field \(k\) has characteristic zero, and let \(C \subseteq \mathbb{P}^N, N \geq 3\), be an integral curve which is not contained in a hyperplane. Let \(D\) and \(E\) be two nonempty linear systems on \(C\). A general hyperplane section \(\Gamma\) of \(C\) has the following property:

\[
c(\Gamma, D + E) \geq \min \{ \deg C, c(\Gamma, D) + c(\Gamma, E) - 1 \}
\]

**Proof** The general hyperplane section \(\Gamma\) consists of \(d = \deg C\) distinct smooth points of \(C\), and by the Uniform Position Theorem we can assume that all subsets of \(\Gamma\) consisting of \(m\) points impose the same number of conditions on the linear systems \(D, E\) and \(D + E\).

If \(c(\Gamma, D) = 0\) or \(c(\Gamma, E) = 0\), the claim is obvious. Thus we can assume \(c(\Gamma, D) \geq 1\) and \(c(\Gamma, E) \geq 1\). We first treat the case

\[
d = \deg C \geq c(\Gamma, D) + c(\Gamma, E) - 1
\]

We can choose disjoint subsets \(\Gamma_1\) and \(\Gamma_2\) of \(\Gamma\) and a point \(P \in \Gamma\) which is not in \(\Gamma_1 + \Gamma_2\) such that \(\deg \Gamma_1 = c(\Gamma, D) - 1\) and \(\deg \Gamma_2 = c(\Gamma, E) - 1\). As \(\Gamma_1 + P\) imposes independent conditions on \(D\), there is a divisor \(D' \in D\) which contains \(\Gamma_1\) but not \(P\). Similarly, there is a divisor \(E' \in E\) which contains \(\Gamma_2\) but not \(\Gamma\). By the Uniform Position Theorem, \(\Gamma_1 + \Gamma_2\) imposes independent conditions on \(D + E\), thus

\[
c(\Gamma, D + E) > c(\Gamma_1 + \Gamma_2, D + E) = \deg \Gamma_1 + \deg \Gamma_2 = c(\Gamma, D) + c(\Gamma, E) - 2
\]

Assume now

\[
d \leq c(\Gamma, D) + c(\Gamma, E) - 1.
\]

We can write \(\Gamma = \Gamma_1 + \Gamma_2 + P\) with \(P\) a single point, \(\deg \Gamma_1 < c(\Gamma, D)\) and \(\deg \Gamma_2 < c(\Gamma, E)\). Then by the Uniform Position Theorem \(\Gamma_1 + P\) imposes independent conditions on \(D\), therefore there is a divisor \(D' \in D\) which contains \(\Gamma_1\) but not \(P\). Similarly, there is a divisor \(E' \in E\) which contains
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Γ₂ but not P. Hence D + E contains Γ₁ + Γ₂ but not Γ. By the Uniform Position Theorem we have

\[ c(Γ, D + E) > c(Γ₁ + Γ₂, D + E) = d - 1 \]

Example 3.3.4 An integral curve \( C \subseteq \mathbb{P}^N \) comes together with several linear systems: we’ll denote by \( \mathcal{E}_n \) the linear system cut out by hypersurfaces of degree \( n \) in \( \mathbb{P}^N \), and by \( |\mathcal{O}_C(n)| \) the complete linear series associated to it: we will also consider \( |\mathcal{O}_C(n)| + |\mathcal{O}_C(m)| \), which is a sub-system of \( |\mathcal{O}_C(n + m)| \).

If \( Γ = C \cap H \) is a hyperplane section of \( C \) consisting of distinct smooth points, we have

\[ |\mathcal{O}_C(n)| - Γ \sim |\mathcal{O}_C(n - 1)| \]

because \( |\mathcal{O}_C(n)| \) is complete, therefore

\[ c(Γ, |\mathcal{O}_C(n)|) = \dim |\mathcal{O}_C(n)| - \dim |\mathcal{O}_C(n - 1)| = \partial h^0 \mathcal{O}_C(n). \]

For the linear systems \( \mathcal{E}_n \) we have \( \dim \mathcal{E}_n = h^0(\mathbb{P}^N, \mathcal{O}(n)) - h^0(\mathbb{P}^N, \mathcal{J}_C(n)) \) and \( \dim \mathcal{E}_n - Γ = h^0(\mathbb{P}^N, \mathcal{J}_{Γ, \mathbb{P}^N}(n)) - h^0(\mathbb{P}^N, \mathcal{J}_C(n)) \); thus

\[ c(Γ, \mathcal{E}_n) = h^0(\mathbb{P}^N, \mathcal{O}(n)) - h^0(\mathbb{P}^N, \mathcal{J}_{Γ, \mathbb{P}^N}(n)) \]

\[ = h^0(H, \mathcal{O}_H(n)) - h^0(H, \mathcal{J}_{Γ, H}(n)) \]

is just the Hilbert function of \( Γ \).

Stanley in [48] proves the following Theorem for integral ACM subschemes of \( \mathbb{P}^N \) over a field of characteristic zero. His proof works as well for integral quasi ACM subschemes:

**Theorem 3.3.5 (Stanley)** Assume the ground field \( k \) has characteristic zero. Let \( X \subseteq \mathbb{P}^N \) be an integral quasi ACM subscheme of dimension \( t \geq 1 \) with index of speciality \( e = e(X) \).

For all integers \( m, n \) satisfying \( m \geq 0, n \geq 1 \) and \( m + n < e + t + 1 \) we have

\[ \sum_{k=1}^{n} h_X(k) \leq \sum_{k=1}^{n} h_X(m + k). \]

**Remark 3.3.6** A sequence \( h(n) \) satisfying the conditions of Theorem 3.3.5 does not necessarily arise from the spectrum of an integral curve. For example, there are no integral curves with spectrum

\[ \{0, 1^2, 2^3, 3^2, 4^2\} \]

Indeed, suppose \( C \) is an integral curve with the above spectrum. Then \( h^0\mathcal{O}_C(1) = 4 \), so that \( C \) is a nondegenerate curve in \( \mathbb{P}^3 \), and \( h^0\mathcal{O}_C(3) = 18 \),
so $s(C) \leq 3$. The biliaison trick shows that there is no such curve on an integral quadric or cubic surface in $\mathbb{P}^3$, hence there is no such curve $C$.

Note that in this example, the sequence $h(n)$ is a 3-sequence which is not of decreasing type. Thus Theorem 3.3.5 doesn’t imply the result of Gruson-Peskine for integral ACM curves in $\mathbb{P}^3$.

**Remark 3.3.7** Theorem 3.3.5 implies Corollary 3.2.2 for an integral curve $C$.

**Proof of Theorem 3.3.5** We may assume that $t = 1$: indeed, if $t \geq 2$, we can cut $X$ with a general hyperplane $H$; by Bertini’s Theorem the intersection $Y$ of $X$ and $H$ is an integral scheme, and by Corollary 1.2.7 $Y$ is quasi ACM and $sp_Y = sp_X$; hence $e(Y) = e(X) - 1$, and the Theorem holds for $X$ if and only if it holds for $Y$.

So let $X = C$ be a curve in $\mathbb{P}^N$. Choose integers $m$ and $n$ satisfying $m \geq 0$, $n \geq 1$ and $m + n < e + 2$. We may assume that the embedding $C \subseteq \mathbb{P}^N$ is defined by the complete linear system $|O_C(1)|$ since the spectrum of $C$ depends only on the integers $h^0O_C(n)$. Then $C$ is a nondegenerate integral curve in $\mathbb{P}^N$ with $N = h^0O_C(1) - 1$. If $N = 1, 2$ the statement is obvious, so we can also assume $N \geq 3$. By Corollary 3.3.3, the general hyperplane section $\Gamma$ of $C$ consists of distinct smooth points in general positions, and

$$c(\Gamma, |O_C(m)| + |O_C(n)|) \geq \min (\deg C, c(\Gamma, |O_C(m)|) + c(\Gamma, |O_C(n)|) - 1)$$

On the other hand, as $|O_C(m)| + |O_C(n)|$ is a linear sub-system of $|O_C(m + n)|$, we have $c(\Gamma, |O_C(m + n)|) \geq c(\Gamma, |O_C(m)| + |O_C(n)|)$. Now recall that

$$c(\Gamma, |O_C(n)|) = \partial h^0O_C(n) = \sum_{k=0}^{n} h_C(k)$$

$$\deg C = \sum_{k=0}^{e+2} h_C(k)$$

$$h_C(0) = 1$$

Hence the above inequalities give

$$\sum_{k=0}^{m+n} h_C(k) \geq \min (\sum_{k=0}^{e+2} h_C(k), \sum_{k=0}^{m} h_C(k) + \sum_{k=1}^{n} h_C(k))$$

As $m + n < e + 2$, subtracting $\sum_{k=0}^{m} h_C(k)$ from both sides we obtain

$$\sum_{k=m+1}^{m+n} h_C(k) \geq \sum_{k=1}^{n} h_C(k)$$
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which is what we had to prove.

Remark 3.3.8 Applying the Theorem to an integral curve $C$ with $n = 1$ we obtain:

$$h_C(1) \leq h_C(l) \quad \text{if } 1 \leq l \leq e(C) + 1$$

In fact, the proof gives a stronger result, which is in in [1], Exercises to Chapter 3, and which we now explain. The point is that we threw out some information by using the inequality $c(\Gamma, |O_C(m + n)|) \geq c(\Gamma, |O_C(m)| + |O_C(n)|)$. When $n = 1$, it is easy to be more careful: the map

$$|O_C(m)| + |O_C(1)| - \Gamma \longrightarrow |O_C(m)|$$

defined by $D \mapsto D - \Gamma$ is an isomorphism (its inverse being $E \mapsto E + \Gamma$!), thus

$$\dim (|O_C(m)| + |O_C(1)| - \Gamma) = \dim |O_C(m)|$$

On the other hand, if $\mu_C(l)$ denotes the number of minimal generators in degree $l$ of the Rao module $\oplus_m H^1(P^N, J_C(n))$ over the coordinate ring of $P^N$, we have

$$\mu_C(m + 1) = \dim |O_C(m + 1)| - \dim (|O_C(m)| + |O_C(1)|)$$

defined by $D \mapsto D - \Gamma$ is an isomorphism (its inverse being $E \mapsto E + \Gamma$!), thus

$$\dim (|O_C(m)| + |O_C(1)| - \Gamma) = \dim |O_C(m)|$$

$$c(\Gamma, |O_C(m)| + |O_C(1)|) = \partial h^0 O_C(m + 1) - \mu_C(m + 1).$$

As in the proof of the Theorem, this gives the inequality

$$h_C(l) \geq h_C(1) + \mu_C(l) \quad \text{if } 1 \leq l \leq e + 1$$

The same trick implies that either $\mu_C(e + 2) = 0$ or

$$h_C(e + 2) \geq \mu_C(e + 2) + h_C(1).$$

It also shows that $\mu_C(l) = 0$ for $l > e + 2$, but this is true for any Cohen Macaulay curve and in any characteristic, for example by Theorem 1.7.1.

Example 3.3.9 In [1], page 117, a close relative of the above argument is used to prove Max Noether's Theorem on the projective normality of a canonical curve. We can translate that proof as follows: let $C \subseteq P^N$ be a canonical curve, that is, an integral curve with $O_C(1) = \omega_C$ and arithmetic genus $g \geq 3$. Then $e(C) = 1$ and the spectrum of $C$ must be

$$\{0, 1^{g-2}, 2^{g-2}, 3\}.$$
3.4 The Spectrum and The Postulation

Hartshorne and Sols ([30], Lemma 2.1) prove that every canonical curve $C$ of degree 8 in $\mathbb{P}^3$ is contained in a cubic surface, so that $C$ is not a curve of maximal rank. In this section we generalize their result. Roughly speaking, the presence of $r + 1$ independent relations between $r$ elements of the Rao module of a curve in $\mathbb{P}^3$ should impose restrictions on the postulation of the curve. In Proposition 3.4.2 below we show that this is indeed the case if the $r \times r$ minors of the matrix expressing the relations have no nontrivial common divisor. But we don’t know how to treat the "degenerate" cases in which these minors do have a common divisor. Of course, when $r = 1$ and the relations are linear, there are no degenerate cases and everything works fine. We then treat in detail the case $r = 2$ with linear relations of the same degree. In particular, we are able to prove that a canonical curve of degree 10 in $\mathbb{P}^3$ lies on a quartic surface. This was already observed by Gruson and Peskine in [15] pp. 56-58, where they study the Hilbert scheme of curves of degree 10 and genus 6. Their argument goes as follows: let $C$ be a canonical curve of degree 10 in $\mathbb{P}^3$. The canonical series embeds $C$ as a curve $D$ in $\mathbb{P}^5$. Using a result of Mumford [37] pp.56-57, which implies that $A_C$ is generated over $k[x, y, z]$ by its elements of degree 1, if $x, y$ and $z$ are sufficiently general linear forms on $\mathbb{P}^5$, they conclude that a general projection of $D$ to $\mathbb{P}^3$ lies on a quartic surface. By semicontinuity, $C$ must also lie on a quartic surface. Another proof can be given using the main Theorem of [25].

We think this approach is interesting because it points (heuristically) to a generalization to non ACM curves of Proposition 2.1 in [15], which implies the connectedness of the numerical character for integral ACM curves. On the other hand, our method is doomed to fail in treating the "degenerate" cases, and perhaps we are being too naive about this all business. Still, we may hope that a different approach will be more successful, and we suggest studying whether the following claim is true or false:

Let $C$ be an integral curve in $\mathbb{P}^3$ with spectrum $\{n^{h_C(n)}\}$. Then the function $n \mapsto h_C(n)$ is strictly increasing to the right of zero until it becomes bigger than or equal to $s(C)$.

The claim is true for integral ACM curves and for nonspecial curves; using the results of this section, one can verify that it is also true for arbitrary integral curves with $s(C) \leq 5$.
3.4. THE SPECTRUM AND THE POSTULATION

3.4.1 The Nondegenerate Case

Let \( R = k[x, y, z, w] \) be the homogeneous coordinate ring of \( \mathbb{P}^3 \). All the \( R \) modules we consider in this section are graded and finitely generated. For any two such modules \( M \) and \( N \), the module \( \text{Hom}_R(M, N) \) is graded, its homogeneous elements of degree \( l \) being the homogeneous homomorphisms of degree \( l \). It follows that the modules \( \text{Ext}_R^i(M, N) \) are graded. All the morphism we will consider will be homogeneous, usually of degree zero, unless otherwise specified.

**Lemma 3.4.1** Let \( D \) be an arithmetically Cohen-Macaulay curve in \( \mathbb{P}^3 \) and let \( \Omega_D = \text{Ext}_R^2(R_D, R(-4)) \) be the canonical module of \( D \). Let \( J \) be a homogeneous ideal in \( R \). There is an isomorphism of graded \( R \)-modules:

\[
\text{Ext}_R^1(\Omega_D, R/J) \cong \frac{J \cap I_D}{J \cdot I_D}(4)
\]

**Proof** Consider a free graded resolution of \( R_D \) over \( R \):

\[
0 \to F_2 \xrightarrow{\beta} F_1 \xrightarrow{\alpha} R \to R_D \to 0 \tag{3.4}
\]

Dualizing and twisting by \( R(-4) \) we obtain a resolution of \( \Omega_D \):

\[
0 \to R(-4) \xrightarrow{\alpha^\vee} F_1^\vee(-4) \xrightarrow{\beta^\vee} F_2^\vee(-4) \to \Omega_D \to 0 \tag{3.5}
\]

We compute \( \text{Ext}_R^1(\Omega_D, R/J) \) applying the functor \( \text{Hom}_R(-, R/J) \) to the sequence 3.5. For a free graded module \( F \) we have \( \text{Hom}_R(F^\vee, R/J) \cong R/J \otimes F \), thus we find

\[
\text{Ext}_R^1(\Omega_D, R/J) \cong \frac{\ker (1 \otimes \alpha(4) : R/J \otimes F_1(4) \to R/J \otimes R(4))}{\text{Im} (1 \otimes \beta(4) : R/J \otimes F_2(4) \to R/J \otimes F_1(4))}
\]

This together with free resolution 3.4 gives

\[
\text{Ext}_R^1(\Omega_D, R/J) \cong \text{Tor}_1^R(R/J, R_D)(4) \cong \frac{J \cap I_D}{J \cdot I_D}(4)
\]

(the last isomorphism being a standard exercise in homological algebra).

**Proposition 3.4.2** Let \( C \) and \( D \) be two curves in \( \mathbb{P}^3 \) and assume that \( D \) is arithmetically Cohen-Macaulay. Suppose there is a nonzero homogeneous homomorphism of degree \( l \)

\[
\phi : \Omega_D \to M_C(l)
\]

from the canonical module of \( D \) to the Rao module of \( C \). Then either \( C \) and \( D \) have a common component, or there is a surface of degree \( l + 4 \) which contains both \( C \) and \( D \) and whose equation is not in \( I_C \cdot I_D \).
Proof We apply the functor $\text{Hom}_R(\Omega_D, -)$ to the exact sequence

$$0 \rightarrow R_C \rightarrow A_C \rightarrow M_C \rightarrow 0$$

to obtain the exact sequence:

$$\text{Hom}_R(\Omega_D, A_C) \rightarrow \text{Hom}_R(\Omega_D, M_C) \rightarrow \text{Ext}^1_R(\Omega_D, R_C).$$

By assumption the group in the middle is nonzero in degree $l$, so at least one of the other two groups is nonzero in degree $l$. If $\text{Hom}_R(\Omega_D, A_C)$ is nonzero, then $C$ and $D$ have a common component because:

$$\text{Supp}(\text{Hom}_R(\Omega_D, A_C)) = \text{Ass}(A_C) \cap \text{Supp}(\Omega_D) = \text{Ass}(C) \cap \text{Ass}(D).$$

On the other hand, assume that $\text{Ext}^1_R(\Omega_D, R_C)$ is nonzero in degree $l$. Then by Lemma 3.4.1

$$\frac{I_C \cap I_D}{I_C \cdot I_D}$$

is nonzero in degree $l + 4$.

Remark We can use the same argument anytime we have an exact sequence

$$0 \rightarrow R_C \rightarrow B \rightarrow N \rightarrow 0,$$

where $B$ is an $R$ module with $\text{Ass}(B) = \text{Ass}(R_C)$, together with a nonzero homogeneous homomorphism $\Omega_D \rightarrow N$; the case we have in mind is $B = \Omega_C(n)$.

Corollary 3.4.3 Let $C$ be a curve in $\mathbb{P}^3$. Suppose there is a nonzero $m \in H^1(J_C(t))$ which is killed by two independent linear forms $z$ and $w$. Then either $C$ contains the line $L$ of equations $z = w = 0$, or $C$ lies on a surface of degree $t + 2$.

Proof Mapping 1 to $m$ we get a nonzero morphism of degree $t$ from $R_L$ to $M_C$. Since $\Omega_L \cong R_L(-2)$, the statement follows from Proposition 3.4.2.

Corollary 3.4.4 Let $C$ be a curve in $\mathbb{P}^3$. Suppose that for some integer $t \geq 0$ we have

$$h_C(n) = \begin{cases} 
    n + 1 & \text{if } -1 \leq n \leq t - 1 \\
    t + 2 & \text{if } n = t, t + 1 
\end{cases}$$

Then either $C$ contains a line in its support or $s(C) \leq t + 2$. 
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**Proof** Assume \( s(C) \geq t + 2 \). Then our assumption on the spectrum implies that \( h^1J_C(t) = 1 \) and \( h^1J_C(t + 1) = 2 \); thus there is a nonzero element \( m \in H^1J_C(t) \) which is killed by two linearly independent linear forms. Now the statement follows from Corollary 3.4.3.

**Remark** Using the Remark following Proposition 3.4.2 with \( B = \Omega_C(-e) \), one can prove variants of the previous Corollary. For example, one can assume

\[
h_C(e + 2 - n) = \begin{cases} 
    n + 1 & \text{if } 0 \leq n \leq t - 1 \\
    t + 2 & \text{if } n = t, t + 1
\end{cases}
\]

and still conclude that either \( C \) contains a line in its support, or \( s(C) \leq t + 2 \).

**Example** Suppose \( C \) is an integral curve of degree 8 satisfying \( \omega_C \cong O_C(1) \). Then \( e(C) = 1 \) and by duality \( h_C(l) = h_C(3 - l) \) for all integers \( l \). Hence the spectrum of \( C \) is

\[
\{0, 1^3, 2^3, 3\}
\]

thus \( C \) satisfies the hypothesis of the Corollary with \( t = 1 \). As \( C \) is integral, we conclude that \( C \) is contained in a surface of degree three. Note that \( h^0O_C(3) = 20 \), thus there is no such curve of maximal rank in \( \mathbb{P}^3 \). This was already proven by Hartshorne and Sols in \([30]\), Lemma 1.2, by a similar method.

We can construct a curve \( C \) with the same spectrum, but not lying on any cubic surface as follows: let \( D \) be the disjoint union of a line and a twisted cubic curve and let \( C \) be the curve obtained from \( D \) by an elementary double link of height one on a quartic surface.

**Corollary 3.4.5** Let \( C \) be a curve in \( \mathbb{P}^3 \). Suppose there is a nonzero \( m \in H^1J_C(t) \) which is killed by three linearly independent linear forms. Then \( s(C) \leq t + 2 \).

**Proof** For each line \( L \) whose equations kill \( m \) we get a morphism of degree \( t \) from \( R_L \) to \( M_C \), so either \( C \) contains \( L \) or \( s(C) \leq t + 2 \). By assumption, there is a one dimensional family of such lines, and \( C \) cannot contain all of them. Hence \( s(C) \leq t + 2 \).

3.4.2 An Eagon-Northcott Complex

In this section we consider a degree zero homomorphism

\[
\phi : F_2 \longrightarrow F_1
\]

of free finitely generated graded \( R \)-modules, satisfying \( \text{rank} F_1 = \text{rank} F_2 + 1 \).

We let \( r = \text{rank} F_2 \) and \( \delta = c_1F_1 - c_1F_2 \). There is a standard way to
construct a complex:

\[ F_2 \xrightarrow{\phi} F_1 \xrightarrow{\psi} R(\delta) \tag{3.6} \]

where the map \( \psi \) is defined as follows: if \( L \) is a free graded \( R \) module of rank \( t \), the natural pairing

\[ \bigwedge^n L \times \bigwedge^{t-n} L \to \bigwedge^t L \cong R(c_1 L) \]

induces an isomorphism \( \bigwedge^n L \cong (\bigwedge^{t-n} L)^\vee (c_1 L) \). If we identify \( F_1 \) with \( (\bigwedge^r F_1)^\vee (c_1 F_1) \) and \( R(\delta) \) with \( (\bigwedge^r F_2)^\vee (c_1 F_1) \), then

\[ \psi = \bigwedge^r \phi^\vee (c_1 F_1). \]

For \( v \in F_1 \) and \( w_1 \wedge \cdots \wedge w_r \in \bigwedge^r F_2 \), we have

\[ \psi(v)(w_1 \wedge \cdots \wedge w_r) = v \wedge \phi(w_1) \wedge \cdots \wedge \phi(w_r) \]

from which we see that \( \psi \circ \phi = 0 \), that is, (3.6) is a complex, and that, fixed some bases, the image of \( \psi \) is the ideal generated by the \( r \times r \) minors of \( \phi \) twisted by \( \delta \). The following Proposition describes what happens dualizing (3.6) when \( \phi \) is injective.

**Proposition 3.4.6** Let \( F_1 \) and \( F_2 \) be free finitely generated graded \( R \)-modules of rank \( r+1 \) and \( r \) respectively; let \( \delta = c_1 F_1 - c_1 F_2 \). Suppose we have an exact sequence of graded \( R \) modules:

\[ 0 \to F_2 \xrightarrow{\phi} F_1 \to Q \to 0 \tag{3.7} \]

If \( d \) is the degree of the greatest common divisor of the \( r \times r \) minors of \( \phi \), a free resolution for the cokernel \( \text{Ext}^1_R(Q, R) \) of \( \phi^\vee \) has the form:

\[ 0 \to R(d - \delta) \to F_1^\vee \xrightarrow{\phi^\vee} F_2^\vee \to \text{Ext}^1_R(Q, R) \to 0 \tag{3.8} \]

**Proof** Dualizing (3.7) we obtain an exact sequence

\[ 0 \to Q^\vee \to F_1^\vee \xrightarrow{\phi^\vee} F_2^\vee \to \text{Ext}^1_R(Q, R) \to 0 \tag{3.9} \]

so it is enough to show that

\[ Q^\vee \cong R(d - \delta) \]

Let \( I \) be the ideal generated by the \( r \times r \) minors of \( \phi \); \( I(\delta) \) is the image of the maps \( \psi \) in (3.6). Note that \( I \) is nonzero because \( \phi \) is injective. If
3.4. THE SPECTRUM AND THE POSTULATION

\( d \) is the degree of the greatest common divisor of the \( r \times r \) minors of \( \phi \), then \( I \cong J(-d) \) where \( J \) is a homogeneous ideal defining a subscheme of codimension at least 2 in \( \mathbb{P}^3 \) (possibly \( J = R \)). Then \( \operatorname{Ext}^i_R(R/J, R) = 0 \) for \( i = 0, 1 \), hence \( J^\vee \cong R \) and we conclude:

\[
I^\vee \cong R(d)
\]

On the other hand, since (3.6) is a complex, there is a surjective map

\[
Q \to I(\delta) \to 0
\]

whose kernel \( N \) is a torsion module, because both \( Q \) and \( I \) have rank one. Hence \( N^\vee = 0 \) and we find

\[
Q^\vee \cong I(\delta)^\vee \cong R(d - \delta)
\]

which concludes the proof.

3.4.3 Analysis of a Special Case

We specialize the situation of the previous section to the case \( F_2 = R^2(-3) \) and \( F_1 = R^3(-2) \), where we have \( r = 2 \) and \( \delta = 0 \). We assume given an injective map

\[
\phi : R^2(-3) \to R^3(-2)
\]

with cokernel \( Q \). To simplify the notation we let

\[
\Omega = \operatorname{Ext}^1_R(Q, R)(-4)
\]

By Proposition 3.4.6 \( \Omega \) has a free resolution of the form

\[
0 \to R(d - 4) \to R^3(-2) \xrightarrow{\phi^\vee} R^2(-1) \to \Omega \to 0 \tag{3.10}
\]

with \( 0 \leq d \leq 2 \).

**Proposition 3.4.7** In the above situation the following are equivalent:

- (3.10) is a minimal resolution of \( \Omega \).
- \( d \) is either 0 or 1
- The elements of degree 2 in \( \Omega \) form a vector space of dimension 5.

**Proof** clear.

The following Proposition describes \( \Omega \) when \( d \) equals zero or one.
Proposition 3.4.8 Let $d$ be the degree of the greatest common divisors of the $2 \times 2$ minors of $\phi$. If $d = 0$, there is an ACM curve $D$ of degree 3 and genus 0 such that $\Omega \cong \Omega_D$. If $d = 1$, one of the following holds:

- there are a line $L$, a plane $H$ and an exact sequence:
  
  \[ 0 \to \Omega_L(1) \to \Omega \to R_H(-1) \to 0 \]

- there are a plane $H$ and a point $P \in H$ such that $\Omega$ is isomorphic to $I_{P,H}$, the homogeneous ideal of $P$ in $H$.

**Proof** If $d = 0$, it follows from the Hilbert-Burch Theorem that the cokernel $Q$ of $\phi$ is isomorphic to the ideal $I$ generated by the $2 \times 2$ minors of $\phi$, and that $R/I$ is Cohen-Macaulay, hence the coordinate ring of an ACM curve $D$ in $\mathbb{P}^3$. From the resolution of $I$ we see that $D$ has degree 3 and genus 0. By definition we have:

\[ \Omega = \text{Ext}_R^1(Q, R)(-4) \cong \text{Ext}_R^1(I, R)(-4) = \Omega_D \]

Assume now $d = 1$. In this case $I = J(-1)$ where $J$ is generated by linear forms; since $d = 1$, $J$ contains at least two independent linear forms, so $J$ is either the ideal of a line or the ideal of a point. Suppose first that $J = I_L$ is the ideal of a line $L$. We look at the diagram:

\[
\begin{array}{cccccccc}
0 & \to & R^2(-3) & \xrightarrow{\phi} & R^3(-2) & \to & Q & \to & 0 \\
\downarrow & & \downarrow \cong & & \downarrow & & \downarrow & \\
0 & \to & K & \to & R^3(-2) & \xrightarrow{\psi} & I_L(-1) & \to & 0
\end{array}
\]

from which we see that $K \cong R(-3) \bigoplus R(-2)$; if we let $N$ denote the cokernel of the first vertical arrow $R^2(-3) \to K$, then $N$ has a resolution

\[ 0 \to R(-3) \to R(-2) \to N \to 0 \]

so $N \cong R_H(-2)$ for some plane $H$; now the Snake’s Lemma gives an exact sequence:

\[ 0 \to R_H(-2) \to Q \to I_L(-1) \to 0. \]

Dualizing we get

\[ 0 \to \text{Ext}_R^1(I_L(-1), R) \to \text{Ext}_R^1(Q, R) \to \text{Ext}_R^1(R_H(-2), R) \to 0 \]

Since $\text{Ext}_R^1(I_L(-1), R) \cong \Omega_L(5)$ and $\text{Ext}_R^1(R_H, R) \cong R_H(1)$, twisting by $R(-4)$ we obtain the desired sequence:

\[ 0 \to \Omega_L(1) \to \Omega \to R_H(-1) \to 0 \]
Assume now that $J = I_P$ is the ideal of a point $P$. We look at the diagram:

\[
\begin{array}{ccccccc}
0 & \rightarrow & R^2(-3) & \overset{\varphi}{\rightarrow} & R^3(-2) & \rightarrow & Q & \rightarrow & 0 \\
& & \downarrow{\tau} & \approx & \downarrow & & \\
0 & \rightarrow & K & \rightarrow & R^3(-2) & \overset{\psi}{\rightarrow} & I_P(-1) & \rightarrow & 0
\end{array}
\]

This time the map 

\[\psi : R^3(-2) \rightarrow I_P(-1)\]

is a minimal free presentation and $K$ is a first syzygy in the Koszul resolution of $I_P(-1)$, hence $K$ fits in a sequence:

\[0 \rightarrow R(-4) \rightarrow R^3(-3) \overset{\pi}{\rightarrow} K \rightarrow 0\]

The first vertical arrow $\tau : R^2(-3) \rightarrow K$ in the diagram, which is injective, must factor as $\pi \circ \sigma$ for some injective map

\[\sigma : R^2(-3) \rightarrow R^3(-3)\]

Now the cokernel $N$ of $\tau$ fits into a diagram:

\[
\begin{array}{ccccccc}
0 & 0 & & & & & \\
& & & & & & \\
& & R(-4) & \rightarrow & R(-4) & & \\
& & & & & & \\
0 & \rightarrow & R^2(-3) & \overset{\sigma}{\rightarrow} & R^3(-3) & \rightarrow & R(-3) & \rightarrow & 0 \\
& & \approx & \downarrow{\pi} & \downarrow & & \\
0 & \rightarrow & R^2(-3) & \overset{\tau}{\rightarrow} & K & \rightarrow & N & \rightarrow & 0 \\
& & & & \downarrow & \downarrow & & \\
& & & & 0 & 0 & &
\end{array}
\]

Thus $N \cong R_H(-3)$ for some plane $H$. The Snake’s Lemma applied to our original diagram gives an exact sequence:

\[0 \rightarrow R_H(-3) \rightarrow Q \rightarrow I_P(-1) \rightarrow 0.\]

Now $\text{Ext}^1_R(I_P, R) = 0$ and $\text{Ext}^2_R(Q, R) = 0$ (because $Q$ has projective dimension at most one), while $\text{Ext}^1_R(R_H, R) \cong R_H(1)$ and $\text{Ext}^2_R(I_P(-1), R) \cong R_P(4)$. Hence dualizing the above sequence and twisting by $R(-4)$ we get:

\[0 \rightarrow \Omega \rightarrow R_H \rightarrow R_P \rightarrow 0.\]

which shows that the point $P$ lies on the plane $H$ and that $\Omega$ is isomorphic to the homogeneous ideal of $P$ in $H$. 

3.4. The spectrum and the postulation
3.4.4 A Canonical Curve of Degree 10 in $\mathbb{P}^3$ lies on a Quartic Surface

**Proposition 3.4.9** Let $C$ be an integral curve in $\mathbb{P}^3$ whose spectrum satisfies

$$h_C(1) = h_C(2) = 4$$

Then $C$ lies on a quartic surface.

**Proof** Let $C$ be an integral curve with $h_C(1) = h_C(2) = 4$. We may assume that $C$ is not contained in any quadric surface (in fact, one can easily show that there is no such curve lying on a quadric surface). The assumption on the spectrum then implies that the cohomology of $C$ in low degrees looks as follows:

<table>
<thead>
<tr>
<th>$l$</th>
<th>$H^0\mathcal{J}_C(l)$</th>
<th>$H^0\mathcal{O}_{\mathbb{P}^3}(l)$</th>
<th>$H^0\mathcal{O}_C(l)$</th>
<th>$H^1\mathcal{J}_C(l)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>4</td>
<td>6</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>10</td>
<td>15</td>
<td>5</td>
</tr>
</tbody>
</table>

In particular, there are two generators of the Rao module in degree 1 with at least three independent linear relations among them. Hence there is a graded $R$-module $\Omega$ with presentation

$$R^3(-2) \xrightarrow{\sigma} R^2(-1) \rightarrow \Omega$$

and a homogeneous morphism $\tau : \Omega \rightarrow M_C$ which is an isomorphism in degree one; furthermore, the degree 2 component of $\Omega$ has dimension 5.

Now let

$$\phi = \sigma^\vee(-4) : R^2(-3) \rightarrow R^3(-2)$$

If $\phi$ is injective, we can use the results of the previous section: by Proposition 3.4.7 the degree of the greatest common divisor of the $2 \times 2$ minors of $\phi$ is either zero or one, and $\Omega$ is described by Proposition 3.4.8. If $d = 0$, then $\Omega \cong \Omega_D$ for some ACM curve $D$ of degree three. Since there is a non-trivial morphism of degree zero from $\Omega$ to the Rao module of $C$, Proposition 3.4.2 implies that $C$ is contained in a quartic surface. If $d = 1$, according to Proposition 3.4.8 we have two cases. In the first one, $\Omega$ fits in an exact sequence

$$0 \rightarrow \Omega_L(1) \rightarrow \Omega \rightarrow R_H(-1) \rightarrow 0$$

The composite map $\Omega_L(1) \rightarrow \Omega \rightarrow M_C$ must be non-zero, otherwise $\tau$ would have to factor through $R_H(-1) \rightarrow M_C$ and it could not be an isomorphism in degree 1. Thus we have a non-zero map $\Omega_L(1) \rightarrow M_C$ and by
3.4. THE SPECTRUM AND THE POSTULATION

Proposition 3.4.2 $C$ lies on a cubic surface. In the second case with $d = 1$, $\Omega$ is isomorphic to the homogeneous ideal of a point $P$ in a plane $H$. In particular, the equation of $H$ kills $\Omega$, hence it kills $H^1(J_C(1))$ as well. If $Z$ is the scheme theoretic intersection of $C$ and $H$, it follows that

$$H^1(H, J_Z(2)) = 2$$

Then the spectrum of $Z$ must be $\{0, 1^2, 2, \ldots, \deg(C) - 1\}$ so that $C$ has a $\deg(C) - 1$ secant line $L$. Now the pencil of planes through $L$ defines a birational map from $C$ to $\mathbb{P}^1$, so that $C$ is a rational curve. In fact, $C$ must be smooth, for example by a Theorem of Gruson-Lazarsfeld-Peskine (see [14] Remark 1 on Page 501). But then the arithmetic genus of $C$ is zero, which is impossible because our assumption on the spectrum implies $g(C) \geq 4$. We have only one more case to treat, when $\phi$ is not injective. In this case all the $2 \times 2$ minors of $\phi$ vanish. So let

$$A = \begin{bmatrix} l_1 & l_2 \\ l_3 & l_4 \end{bmatrix}$$

be a $2 \times 2$ submatrix of $\phi$, where the $l_j$'s are linear forms giving relations

$$l_1 \alpha + l_2 \beta = 0$$
$$l_3 \alpha + l_4 \beta = 0$$

on the two generators $\alpha$ and $\beta$ of $\Omega$. These two relations are linearly independent by our assumption on $\Omega$ and yet the determinant of $A$ must be zero. This implies that there is a nonzero constant $u$ such that $l_2 = u \cdot l_1$ and $l_4 = u \cdot l_3$, while $l_1$ and $l_3$ are independent linear forms. We conclude that the nonzero element $\alpha + u \cdot \beta$ of $\Omega$ is killed by two independent linear forms. Hence, if $L$ is the line $l_1 = l_3 = 0$, there is a nonzero map of degree one from $R_L$ to $\Omega$ and thus to the Rao module of $C$. By Corollary 3.4.3 this implies that $C$ is contained in a cubic surface, and we are done.

Remark We would like to generalize the Proposition for curves satisfying

$$h_C(n) = \begin{cases} n + 1 & \text{if } 0 \leq n \leq t - 1 \\ t + 3 & \text{if } n = t, t + 1 \end{cases}$$

for some $t \geq 1$, the case $t = 1$ being Proposition 3.4.9. Indeed, our proof works for any $t$ except for the case $\Omega = I_{P,H}$.

Example Suppose $C$ is an integral curve of degree 10 satisfying $\omega_C \cong \mathcal{O}_C(1)$. Then $e(C) = 1$ and by duality $h_C(l) = h_C(3 - l)$ for all integers $l$. Hence the spectrum of $C$ is

$$\{0, 1^4, 2^4, 3\}$$
thus $C$ satisfies the hypothesis of the Proposition and we conclude that $C$

is contained in a quartic surface.

**Remark** Suppose $C$ is as in the statement of Proposition 3.4.9 and that 
$\deg C \geq 11$. Then the complete linear series $|O_C(1)|$ embeds $C$ in $\mathbb{P}^5$ as 
a curve $D$ of degree $\geq 11$, which is not contained in any hyperplane, and 
such that

$$h^0(O_D(2)) = 15$$

Now a Theorem of Castelnuovo-Harris ([1] p.120) allows us to conclude (at 
least in characteristic zero) that $D$ lies on a surface of degree 4 in $\mathbb{P}^5$, and 
this of course implies that $C$ lies on a quartic surface in $\mathbb{P}^3$. Furthermore, 
four is the minimal degree possible for an integral nondegenerate surface $S$ 
in $\mathbb{P}^5$. There is a classification of such surfaces (see for example [1] p.96): $S$ 
is either a rational normal scroll (possibly singular) or the Veronese surface, 
and we think it should be easy to write down explicitly the list of all 
possible spectra for integral curves on such surfaces.

### 3.5 An Example: the Spectra of Curves on a 
Ruled Cubic Surface in $\mathbb{P}^3$.

In this section we compute the spectrum of integral curves on a general 
ruled cubic surface $X$ in $\mathbb{P}^3$. This surface has been extensively studied by 
Hartshorne [26], and his results make our computation quite easy.

In [26] Hartshorne develops a theory of generalized divisors on schemes 
$X$ which are locally Gorenstein in codimension zero and one, and which 
satisfy Serre’s condition $S_2$; in brief, we say that such a scheme satisfies 
$G_1 + S_2$. For example, if $X$ is a locally Gorenstein scheme, or a normal 
scheme, $X$ satisfies $G_1 + S_2$.

By [26], Proposition 2.4, if $X \subseteq \mathbb{P}^N$ is a surface satisfying $G_1 + S_2$ (e.g., 
any surface in $\mathbb{P}^3$, or a normal surface in a higher dimensional projective 
space), the effective generalized divisors are in one-to-one correspondence 
with curves $C$ contained in $X$ (where, as always in this dissertation, curve 
means a closed subscheme of pure dimension one with no embedded points). 
There is a well defined notion of linear equivalence between generalized 
divisors, and, in particular, two curves on a surface $X$ satisfying $G_1 + S_2$ are 
linearly equivalent if and only if their ideal sheaves in $O_X$ are isomorphic 
as $O_X$-modules.

From now on, we assume that $X \subseteq \mathbb{P}^N$ is a surface which is ACM and 
satisfies $G_1 + S_2$; for example, $X$ can be an arbitrary surface in $\mathbb{P}^3$, or a
3.5. AN EXAMPLE: THE SPECTRA OF CURVES ON A RULED CUBIC SURFACE IN $\mathbb{P}^3$.

A rational normal scroll in $\mathbb{P}^N$ (see [1] pp.96-99), or a cone in $\mathbb{P}^4$ over a smooth ACM curve in $\mathbb{P}^3$.

Two linearly equivalent curves on $X$ have the same postulation, the same spectrum and isomorphic Rao modules; furthermore, by Proposition 2.2.1, if we know the spectrum of a curve $C$ on $X$ and the spectrum of the hyperplane section $H$, we can compute the spectrum of any curve which is linearly equivalent to $C + nH$ for some integer $n$. Thus, in computing the spectrum, we can restrict our attention to curves $C$ such that $C - H$ is not effective (and nonzero). This motivates the following definition:

**Definition 3.5.1** Let $X \subseteq \mathbb{P}^N$ be a surface which is ACM and satisfies $G_1 + S_2$, and let $H$ denote the hyperplane section of $X$. A curve $C \subseteq X$ is minimal relative to $X$ ($X$-minimal for short) if there is no curve $D$ on $X$ linearly equivalent to $C - H$.

Note that the hyperplane section $H$ is $X$-minimal by definition, as we don’t allow a curve to be empty.

The only $\mathbb{P}^2$-minimal curves are the lines. If $X = Q$, the smooth quadric in $\mathbb{P}^3$, the $Q$-minimal curves are the plane sections and the curves of type either $(a,0)$ or $(0,a)$, with $a \geq 1$. When $X$ is the union of two distinct planes, or a double plane, we have described the spectra of the $X$-minimal curves in Section 2.2.

We now consider the case of the ruled cubic surface. If $S$ is a a rational cubic scroll in $\mathbb{P}^4$, a general projection from a point $Q \in \mathbb{P}^4$ gives a surjective morphism $f : S \to X$, where $X \subseteq \mathbb{P}^3$ is a ruled cubic surface with a double line $L$. As an abstract surface, $S$ is isomorphic to $\mathbb{P}^2$ blown up at one point $P$, and the embedding in $\mathbb{P}^4$ is defined by the linear systems of conics in $\mathbb{P}^2$ passing through $P$. The surface $X$ is studied extensively in Section 6 of [26]; we use the terminology and notation of that section freely, and we recall some of Hartshorne’s results.

We denote by $\Gamma$ the preimage of the double line $L$ in $S$: $\Gamma$ is the strict transform of a line in $\mathbb{P}^2$ not passing through $P$. We let $E$ be the image in $X$ of the exceptional divisor of $S$. The rulings of $X$ are the images of the strict transform of lines in $\mathbb{P}^2$ passing through $P$, and they meet in pairs at points of $L$.

Since $X$ is smooth away from $L$, an integral curve on $X$ is either the line $L$ or an almost Cartier divisor. Thus to compute the possible spectra of integral curves on $X$ it is enough to describe the $X$-minimal curves which are almost Cartier divisors and compute their spectra, and then to determine which linear equivalence classes of almost Cartier divisors contain integral curves. With the exception of the computation of the spectra, this program is carried out in [26], where Hartshorne studies the group $\text{A Pic } X$ of linear equivalence classes of almost Cartier divisors on $X$. 

Hartshorne shows that an element of the almost Picard group \( \text{A Pic} \ X \) can be represented uniquely by a triple \((a, b, \alpha)\), where \(a\) and \(b\) are integers, and \(\alpha \in \text{Div}\Gamma / f^{\ast}\text{Div} \ L\) is such that \(a\) is congruent to \(\deg \alpha\) modulo 2; furthermore, any such triple \((a, b, \alpha)\) is associated to some element of \(\text{A Pic} \ X\).

For example, the plane section \(H\) has class \((2, 1, 0)\), \(E\) has class \((0, -1, 0)\), a ruling has class \((1, 1, \lambda)\), where \(\lambda\) is the class of a point in \(\Gamma\) mapping to the intersection of the ruling with the double line \(L\).

If \(\alpha \in \text{Div}\Gamma / f^{\ast}\text{Div} \ L\), Hartshorne defines \(h(\alpha) \geq 0\) to be the least degree of an effective divisor in \(\text{Div}\Gamma\) representing \(\alpha\).

**Proposition 3.5.2 (cf. [26], 6.5 and 6.7.1)** Let \(C\) be an almost Cartier curve in \(X\). Then \(C\) is \(X\)-minimal if and only if one of the following holds:

1. \(C\) has type \((2, 1, 0)\); in this case, \(C\) is a plane section and has spectrum \(\{0, 1, 2\}\).

2. \(C\) has type \((a, a - 1, \alpha)\), where \(a \geq 1\) and \(h(\alpha) = a\); in this case the spectrum of \(C\) is \(\{0, 1^{a-1}\}\), and the divisor class contains a smooth rational curve of degree \(a + 1\).

3. \(C\) has type \((a, a - 1, \alpha)\), where \(a \geq 1\) and \(h(\alpha) = a + 2r\) for some \(r \geq 1\); in this case the spectrum of \(C\) is \(\{1 - r, 0, 1^{a-1}\}\); an example of such a curve is given by a double structure on \(L\) with genus \(-r\) union \(a - 1\) rulings, each meeting the double structure with multiplicity one.

4. \(C\) has type \((a, a, \alpha)\), where \(a \geq 1\) and \(h(\alpha) = a - 2c \geq 0\) for some \(c \geq 0\); in this case the spectrum of \(C\) is \(\{0^{h(\alpha)+c}, 1^{c}\}\); an example of such a curve is given by the union of \(2c\) rulings intersecting in pairs on \(L\) with \(h(\alpha)\) disjoint rulings.

5. \(C\) has type \((2, b, \alpha)\), where \(b \leq 0\) and \(h(\alpha) = 2r + 2\) for some \(r \geq 0\); in this case the spectrum of \(C\) is \(\{1 - r, 0, 1^{2}\}\) if \(b = 0\), and \(\{1 - r, 0, 1^{2}\} \cup \{1 + b, \ldots, -1, 0\}\) if \(b \leq -1\); an example of such a curve is given by the union of a double structure on \(L\) of genus \(-r\) with one ruling and \((1 - b)E\), if the double structure and the ruling meet with multiplicity one.

6. \(C\) has type \((1, b, \alpha)\), where \(b \leq -1\) and \(h(\alpha) = 2r + 1\) for some \(r \geq 0\); in this case the spectrum of \(C\) is \(\{1 - r, 0\} \cup \{1 + b, \ldots, -1, 0\}\); an example of such a curve is given by the disjoint union of a double structure on \(L\) with genus \(-r\) and \(-bE\).

7. \(C\) has type \((0, b, 0)\), where \(b \leq -1\); in this case the spectrum of \(C\) is \(\{1 + b, \ldots, -1, 0\}\), and \(C = -bE\).
3.5. AN EXAMPLE: THE SPECTRA OF CURVES ON A RULED CUBIC SURFACE IN $\mathbb{P}^3$.

Sketch of the Proof We will not include a complete proof of this Proposition. To determine the $X$-minimal curves, one uses [26], Proposition 6.5. The computation of the spectra of the curves in 3 and 5 is nontrivial: suppose $C$ is as in 3, that is, a double structure $L_r$ on $L$ with genus $-r$ union $a - 1$ rulings, each meeting the double structure with multiplicity one. If $D$ denotes the union of the $a - 1$ rulings in $C$, we have an exact sequence:

$$0 \rightarrow \mathcal{O}_C \rightarrow \mathcal{O}_{L_r} \oplus \mathcal{O}_D \rightarrow \mathcal{O}_{L_r \cap D} \rightarrow 0.$$ 

Now $L_r \cap D$ is the union of $a - 1$ reduced points, therefore the induced map on global sections

$$H^0(\mathcal{O}_{L_r}(n) \oplus \mathcal{O}_D(n)) \rightarrow H^0(\mathcal{O}_{L_r \cap D}(n))$$

is the zero map for $n < 0$, while it is surjective for $n \geq 0$. It follows that $h^0\mathcal{O}_C(n) = h^0\mathcal{O}_{L_r}(n)$ for $n < 0$, while for $n \geq 0$ we have

$$h^0\mathcal{O}_C(n) = h^0\mathcal{O}_{L_r}(n) + (a - 1) h^0\mathcal{O}_{\mathbb{P}^1}(n) - (a - 1).$$

Taking second differences we see that the spectrum of $C$ is $\{1 - r, 0, 1^{a - 1}\}$.

If $C$ is as in 5, the situation is slightly more complicated: one way of proceeding is to first compute the spectrum of the union $D$ of $(1-b)E$ with the ruling $M$. The divisor type of $D$ is

$$(0, b - 1, 0) + (1, 1, \lambda) = (1, b, \lambda),$$

hence $D$ is linearly equivalent to the disjoint union of $-bE$ with a double structure on $L$ of type $(1, 0, \lambda)$. This latter is a planar double structure $L_0$, so it has spectrum $\{0, 1\}$, while the spectrum of $-bE$ is $\{1 + b, \ldots, -1, 0\}$ (because $X$ is smooth along $E$). Thus the spectrum of $D$ is equal to the union of these two spectra:

$$sp_D = \{1 + b, \ldots, -1, 0^2, 1\}.$$ 

One concludes the calculation of the spectrum of $C = D \cup L_r$ by studying as above the exact sequence:

$$0 \rightarrow \mathcal{O}_C \rightarrow \mathcal{O}_{L_r} \oplus \mathcal{O}_D \rightarrow \mathcal{O}_{L_r \cap D} \rightarrow 0$$

where $L_r \cap D$ is a reduced point.

Hartshorne also describes the divisor classes containing integral (resp. smooth) curves:

**Proposition 3.5.3 ([26], Proposition 6.6)** Let $D \in A \text{Pic} X$ be an effective divisor of class $(a, b, \alpha)$. Then:

1. \( D \) is represented by an integral curve if and only if \( D = E \), or \( D \) is a ruling, or \( a > b \geq 0 \) and \( h(\alpha) \leq a \).

2. \( D \) is represented by a smooth curve if and only if \( D = E \), or \( D \) is a union of rulings meeting \( L \) at distinct points, or \( a > b \geq 0 \) and \( h(\alpha) = a \).

### 3.6 Spectra of Integral Curves of Small Degree

In this section we determine (in characteristic zero) all possible spectra of integral curves of degree at most 11 in \( \mathbb{P}^3 \), and we show that they actually occur as spectra of smooth irreducible curves. Later, however, we will show that there is an integral curve of degree 14 in \( \mathbb{P}^3 \) whose spectrum is not the spectrum of any smooth irreducible curve.

More precisely, we find necessary and sufficient conditions for a spectrum to occur as the spectrum of an integral curve of degree at most 11 in \( \mathbb{P}^3 \); in higher degrees these conditions are no longer sufficient, but still necessary. Here we define a spectrum to be a finite set of integers with multiplicities. So if \( sp = \{n^{h(n)}\} \) is a spectrum, the sequence \( h(n) \) is a finite sequence of nonnegative integers.

Given a spectrum \( sp = \{n^{h(n)}\} \), we define its degree by the formula

\[
\deg (sp) = \sum_{n \in \mathbb{Z}} h(n) ,
\]

its genus by the formula

\[
g(sp) = 1 + \sum_{n \in \mathbb{Z}} (n - 1)h(n) ,
\]

and its index of speciality \( e(sp) \) by the formula

\[
e(sp) + 2 = \max\{n : h(n) > 0\} .
\]

Of course, if \( sp \) is the spectrum of a curve \( C \), the degree, genus and index of speciality of \( sp \) coincide with those of \( C \).

**Theorem 3.6.1** Assume char. \( k = 0 \). Let \( C \) be an integral curve in \( \mathbb{P}^N \), and let \( e = e(C) \) be the index of speciality of \( C \). The spectrum \( \{n^{hc(n)}\} \) of \( C \) satisfies the following properties:

1. \( hc(n) > 0 \) if and only if \( 0 \leq n \leq e + 2 \).
3.6. SPECTRA OF INTEGRAL CURVES OF SMALL DEGREE

2. \( h_C(0) = 1 \).

3. the **1-property**: if \( h_C(n) = 1 \) for some \( n \) with \( 1 \leq n \leq e + 1 \), then
   \[
   h_C(n) = 1 \quad \text{for } 1 \leq n \leq e + 2.
   \]

4. the **2-property**: if \( h_C(n) = 2 \) for some \( 2 \leq n \leq e + 1 \), then either
   \[
   h_C(n) = 2 \quad \text{for } 1 \leq n \leq e + 1, \quad \text{or}
   \]
   \[
   n = e + 1 \quad \text{and} \quad h_C(1) = 2, \ h_C(e + 2) = 1.
   \]

5. (cf. Stanley’s Theorem 3.3.5) for all integers \( m, n \) satisfying \( m \geq 0, \ n \geq 1 \) and \( m + n < e + 2 \) we have
   \[
   \sum_{k=1}^{n} h_C(k) \leq \sum_{k=1}^{n} h_C(m + k).
   \]

Conversely, if \( sp \) is a spectrum of degree at most 11 satisfying properties 1 to 5, there is a smooth irreducible curve in \( \mathbb{P}^3 \) with spectrum \( sp \).

**Remark 3.6.2** Note that the 5 properties in the previous theorems are independent; in particular, Stanley’s Theorem 3.3.5 does not imply the 1-property or the 2-property.

Roughly speaking, the 1-property says that 0 and \( e + 2 \) are the only integers \( n \) for which it is natural to have \( h_C(n) = 1 \): if \( h_C(n) = 1 \) for a different value of \( n \), then \( C \) is a plane curve and its spectrum is \( \{0, 1, \ldots, e + 2\} \).

Similarly, the 2-property says that, if \( h_C(n) = 2 \) for some \( n \) satisfying \( 2 \leq n \leq e \), then \( C \) lies on a quadric surface in \( \mathbb{P}^3 \) and its spectrum is:
   \[
   sp_C = \{0, 1^2, 2^2, \ldots, (e + 1)^2, (e + 2)^{d - 2e - 3}\},
   \]
where \( d = \deg(C) \). Also we may have \( e \geq 1 \) and \( h_C(e + 1) = 2 \) without \( C \) being contained in a quadric, but then \( h_C(1) = 2 \), so that \( C \) lies in some \( \mathbb{P}^3 \), and \( h_C(e + 2) = 1 \).

One can wonder at this point what the 3-property might be. If \( C \) is an integral curve in \( \mathbb{P}^3 \), Proposition 2.8.4 tells us that \( h_C(n) = 3 \) for some \( n \) with \( 3 \leq n \leq e - 1 \) forces \( C \) to lie on a cubic surface; however, the possible spectra of curves on integral cubic surfaces in \( \mathbb{P}^3 \) are not as easily described as in the case of the plane or of the quadric surfaces (cf. Section 3.5); also the analysis of the special cases \( h_C(n) = 3 \) for \( n = 2, e, e + 1 \) becomes more involved (cf. the following remark).
Remark 3.6.3 In general, properties 1 to 5 in the theorem are not sufficient to guarantee the existence of an integral curve with the given spectrum. Take for example the spectrum \( sp = \{0, 1^3, 2^3, 3^4, 4^2\} \) which has degree 13 and satisfies properties 1 to 5. Suppose there is an integral curve \( C \) with spectrum \( sp \). The complete linear system \( |\mathcal{O}_C(1)| \) embeds \( C \) in \( \mathbb{P}^4 \) as a nondegenerate curve with \( h^0\mathcal{O}_C(2) = 12 \), hence \( C \) lies on 3 linearly independent quadrics. A classical argument of Castelnuovo (see [1] p.120) implies that \( C \) lies on a surface \( S \) of degree 3; since \( C \) is integral and nondegenerate, \( S \) must be integral. Now the only integral nondegenerate surfaces of degree 3 in \( \mathbb{P}^4 \) are the smooth cubic scroll and the cone over the twisted cubic curve in \( \mathbb{P}^3 \). The Weil divisor class group of the cone is isomorphic to \( \mathbb{Z} \) with generator the class of a ruling \( L \); thus a curve of degree 13 on the cone is in the linear equivalence class of \( L + 4H \), where \( H \) is the hyperplane section, and by Proposition 2.2.1 any curve in this class has spectrum \( \{0, 1^3, 2^1, 3^3, 4^3\} \). Therefore \( C \) cannot lie on the cone.

A similar argument shows that \( C \) cannot lie on the smooth cubic scroll \( S \) either: for example, one can show that, if \( E \) is the exceptional line on \( S \) and if \( F \) is a ruling, then the \( S \)-minimal curves are linearly equivalent to one of the following: the plane section, \( nE, mF \) and \( F + pE \) where \( n, m \) and \( p \) are positive integers. Now an explicit calculation shows that \( sp \) does not occur as the spectrum of a curve on \( S \). Or one can observe that the spectrum of a curve on \( S \) is also the spectrum of some curve on the ruled cubic surface \( X \) of section 3.5, and using Proposition 3.5.2 and the biliaison trick one can prove that \( sp \) does not occur as the spectrum of a curve on \( X \).

Thus there is no integral curve \( C \) with spectrum \( sp = \{0, 1^3, 2^1, 3^3, 4^3\} \). The above argument will give a ”piece of a 3-property”, namely, conditions which must be satisfied by spectra of integral curves \( C \) satisfying \( h_C(1) = h_C(2) = 3 \).

Remark 3.6.4 It is an accident that all spectra of integral curves of degree at most 11 satisfy the following property: for each of them there is an integer \( a \) such that the multiplicity sequence \( h_C(n) \) is nondecreasing to the left of \( a \), and nonincreasing to the right of \( a \). In general, the spectrum of an integral curve may fail to satisfy this property. Take for example a curve of type \((10, 6, \alpha)\) with \( h(\alpha) = 8 \) on the ruled cubic surface of section 3.5: by Proposition 3.5.3 there is such an integral curve, and by Propositions 3.5.2 and 2.2.1 the spectrum of \( C \) is

\[ \{0, 1^2, 2^4, 3^3, 4^4\} \]

In fact, the spectrum of an integral curve may have an arbitrarily large number of ”bumps”: for example, let \( F \) be a smooth quartic surface in \( \mathbb{P}^3 \)
containing a line $L$, and let $H$ be a plane section of $F$. Let $m$ and $n$ be positive integers and look at the divisor $D = mL + nH$. For $n$ very large, $D$ is very ample, so the divisor class of $D$ contains a smooth irreducible curve $C$. The spectrum of $mL$ is

$$\{-2(m-1), -2(m-2), \ldots, -2, 0\}$$

and has $m-1$ "bumps", and for $n$ sufficiently large the spectrum of $D$ has $m-2$ bumps.

**Proof of Theorem 3.6.1**

Let $C$ be an integral curve in $\mathbb{P}^N$. We show now that the spectrum of $C$ satisfies properties 1 to 5.

First of all, since $C$ is integral, we have $h^0(C, \mathcal{O}_C) = 1$, and therefore $h_C(n) = 0$ for $n < 0$ and $h_C(0) = 1$, hence property 2 holds. Property 1 now follows from Theorem 1.7.1. Property 5 is Stanley’s Theorem 3.3.5 for curves.

Suppose now $C$ is an integral nondegenerate curve in $\mathbb{P}^N$ with $N \geq 4$:

$$h_C(1) = N - 1 \geq 3$$

and by Stanley’s Theorem $h_C(n) \geq 3$ for $1 \leq n \leq e + 1$, so that $C$ trivially satisfies the 1-property and the 2-property.

We may therefore assume that $C$ is an integral curve in $\mathbb{P}^3$. The 1-property follows immediately from Theorem 1.7.1 (as well as from Proposition 2.8.4), together with the fact that a plane curve $C$ has spectrum $\{0, 1, \ldots, e + 2\}$.

Thus we only have to prove the 2-property for an integral curve $C \subseteq \mathbb{P}^3$. First assume that $h_C(n) = 2$ for some $n$ satisfying $2 \leq n \leq e + 1$. Then $e \geq 2$, so that $(e + 4)/2 \geq 3$. If we had $s(C) \geq 3$, by Proposition 2.8.4 the spectrum of $C$ would contain the spectrum of the complete intersection $X$ of two surfaces of degree 3 and $e + 1$, therefore we would have

$$h_C(n) \geq h_X(n) \geq 3,$$

a contradiction. Hence $s(C) \leq 2$. On the other hand, $C$ cannot be a plane curve—if it were, it would have $h_C(n) = 1$— hence $C$ must be contained in an integral quadric surface. Now the spectrum of a curve $C$ lying on an integral quadric surface is determined by its degree $d$ and its index of speciality $e$ as follows: if $b = e + 2$ and $a = d - b$, then $C$ has the same cohomology as a divisor of type $(a, b)$ on a smooth quadric surface and the spectrum of $C$ (cf. Section 2.2) is

$$\{0, 1^2, \ldots, (b-1)^2, b^{a-b+1}\}.$$
that is, \( h_C(n) = 2 \) for \( 1 \leq n \leq e + 1 \).

Now suppose that \( h_C(e + 1) = 2 \) and \( e + 1 \geq 2 \). If \( s(C) \leq 2 \), we can argue as in the previous case, so we may assume that \( s = s(C) \geq 3 \).

By Stanley’s Theorem 3.3.5 \( h_C(1) \leq h_C(e + 1) = 2 \), and by the 1-property we must have \( h_C(1) = 2 \). To finish, we have to show that \( h_C(e + 2) = 1 \). We distinguish two cases, as in the proof of Theorem 2.8.2.

First suppose \( s \leq (e + 4)/2 \), and let \( h = e + 4 - s \); we have \( h \geq s \geq 3 \).

By Propositions 2.2.1 and 2.2.3, either \( C \) is the complete intersection of two surfaces of degrees \( s \) and \( h \), in which case \( h_C(e + 2) = 1 \), or \( sp_C = sp_D(-h) \cup sp_X \)

where \( D \) is a curve and \( X \) is a complete intersection of two surfaces of degrees \( s \) and \( h \). Since \( h \geq s \geq 3 \) and \( e(X) = e \), we have \( h_X(e + 1) = 2 \).

Hence

\[
2 = h_C(e + 1) = h_D(e + 1 - h) + h_X(e + 1) = h_D(e + 1 - h) + 2,
\]

so that \( h_D(e + 1 - h) = 0 \).

Now \( e + 1 - h = s - 3 \geq 0 \) and the spectrum of a curve is connected to the right of zero, hence we must have \( h_D(e + 2 - h) = 0 \) (and \( s \geq 4 \)). We conclude that

\[
h_C(e + 2) = h_X(e + 2) = 1,
\]

and this finishes the case \( s \leq (e + 4)/2 \).

Finally, assume that \( s > (e + 4)/2 \) and let \( m \) be the integral part of \( (e + 4)/2 \). Then \( 2m = e + 4 + c_1 \) where \( c_1 = 0 \) or \(-1\), and \( s > m \geq 2 \) because \( e \geq 1 \). A nonzero element of \( H^0\omega_C(-e) \) corresponds to an extension

\[
0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(c_1 - m) \rightarrow \mathcal{F} \rightarrow \mathcal{J}_C(m) \rightarrow 0
\]

where \( \mathcal{F} \) is a rank 2 reflexive sheaf with first Chern class \( c_1 \). \( \mathcal{F} \) is stable because \( s > m \), and by Proposition 2.6.4 we have

\[
sp_C = sp_{\mathcal{F}}(-m) \cup sp_X
\]

where \( X \) is a complete intersection of two surfaces of degrees \( m \) and \( m - c_1 \).

Since \( m - c_1 \geq m \geq 2 \), we see that \( h_{\mathcal{F}}(e + 1 - m) = 0 \) and \( h_C(e + 2) = h_{\mathcal{F}}(e + 2 - m) + 1 \).

Now note that \( e + 1 - m \geq 0 \) because \( m \) is the integral part of \( (e + 4)/2 \) and \( e \geq 1 \); by Theorem 2.5.7, which says in particular that the spectrum of a stable rank 2 reflexive sheaf is connected to the right of zero, we have

\[
h_{\mathcal{F}}(e + 2 - m) = 0
\]

because \( h_{\mathcal{F}}(e + 1 - m) = 0 \). Thus \( h_C(e + 2) = 1 \), and this concludes the proof of the 2-property and of the first half of the theorem.
Before we proceed, we give explicit formulas for spectra which have index of speciality \( e \leq 2 \) and satisfy properties 1 to 5. Let \( sp = \{n^{h(n)}\} \) be a spectrum of degree \( d \) and genus \( g \) and suppose that \( e = e(sp) \leq 2 \). Then \( sp \) satisfies properties 1 to 5 if and only if one of the following holds:

1. \( e = -2 \) and \( sp = \{0\} \).
2. \( e = -1 \) and \( sp = \{0, 1^{d-1}\} \) where \( d \geq 2 \).
3. \( e = 0 \) and either \( sp = \{0, 1, 2\} \), or \( sp = \{0, 1^{d-g-1}, 2^g\} \) where \( g \geq 1 \) and \( d \geq g + 3 \).
4. \( e = 1 \) and either \( sp = \{0, 1, 2, 3\} \), or \( sp = \{0, 1^{a+2}, 2^{b+3}, 3^{c+1}\} \) where \( a, b \) and \( c \) are nonnegative integers with \( b \geq a \).
5. \( e = 2 \) and \( sp = \{0, 1, 2, 3, 4\} \), or \( sp = \{0, 1^2, 2^{2+2}, 3^{2+2}, 4^{2-7}\} \), or 

\[
sp = \{0, 1^{a+2}, 2^{b+3}, 3^{c+2}, 4^{p+1}\}
\]

where \( a, b, c \) and \( p \) are nonnegative integers satisfying \( b \geq a - 1 \), \( c \geq a \), and \( c = 0 \) implies \( p = 0 \).

We must now show that every spectrum of degree at most 11 satisfying properties 1 to 5 occurs as the spectrum of a smooth irreducible curve in \( \mathbb{P}^3 \). We proceed as follows: we fix an order on the set of spectra, then we list in order all spectra of degree at most 11 satisfying properties 1 to 5, and for each spectrum \( sp \) in the list we give an example of a smooth irreducible curve in \( \mathbb{P}^3 \) with spectrum \( sp \).

We actually do a little bit more: if \( sp \) is the spectrum of some integral curve, we define:

\[
s_{\text{max}}(sp) = \max\{s(C) : C \subseteq \mathbb{P}^3 \text{ integral curve with } sp_C = sp\}.
\]

In Tables 3.1-3.6 we list the possible spectra of integral curves of degree \( \leq 11 \) together with their genus and (with 3 exceptions) \( s_{\text{max}} \), and for each spectrum \( sp \) we give an example of a smooth irreducible curve \( C \) with \( sp_C = sp \) and \( s(C) = s_{\text{max}}(sp) \).

We order the set of spectra by letting \( sp_1 > sp_2 \) if \( \deg(sp_1) < \deg(sp_2) \), or if \( \deg(sp_1) = \deg(sp_2) \) and \( g(sp_1) > g(sp_2) \); if \( sp_1 \) and \( sp_2 \) have the same degree and genus, we set \( sp_1 > sp_2 \) if there is an integer \( n \) such that \( h_1(n) > h_2(n) \) and \( h_1(l) = h_2(l) \) for \( l > n \). It follows that, among integral curves of degree \( d \), the plane curves have the largest spectrum \( \{0, \ldots, d-1\} \), while the rational curves have the smallest \( \{0, 1^{d-1}\} \). The semicontinuity theorem implies that if a curve \( C \) specializes in a flat family to a curve \( D \), the spectrum of \( C \) is smaller or equal than that of \( D \) ("the more special
the curve, the bigger the spectrum”). In the tables we list the spectra in decreasing order.

To construct smooth curves with the desired spectra, we have several techniques available. The first one is the explicit study of divisors on surfaces of small degrees: the plane, the smooth quadric surface, the ruled cubic surface of section 3.5, the smooth cubic surface and more generally Del Pezzo surfaces (by which we mean a general projection in $\mathbb{P}^3$ of a smooth Del Pezzo surface).

The second technique we use is that of constructing new curves out of a given one by means of liaison or biliason: the best results in this direction are due to Nollet [39]; they generalize in particular the existence statement in Theorem 3.1.3.

Finally, in the past fifteen years there have been very successful efforts to construct curves with “good cohomological properties”: see [24], Section 4.c, for a review of the first results, and [28], [11], [49] for more recent theorems. As an example, we state the main result of [3], which is the one we use more often in the tables. Recall that a curve $C \subseteq \mathbb{P}^3$ is of maximal rank if, for each integer $n$, the natural map

$$\phi(n) : H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(n)) \to H^0(C, \mathcal{O}_C(n))$$

is of maximal rank, i.e., it is either injective or surjective.

**Theorem 3.6.5** For each pair of integers $(d, g)$ satisfying $d \geq g + 3$ and $g \geq 0$, there exists a nonspecial smooth irreducible curve $C \subseteq \mathbb{P}^3$ of degree $d$ and genus $g$ having maximal rank.

We now explain the notation and terminology we use in the tables. $F_d$ denotes a surface of degree $d$ in $\mathbb{P}^3$, while $Q$ is the smooth quadric surface. $F_3^{\text{ruled}}$ denotes the ruled cubic surface of Section 3.5. We write $C \sim D$ on $F_d$ to mean that the two curves $C$ and $D$ are linearly equivalent on the surface $F_d$, in the sense of [26]. We use the notation of [20] for linear equivalence classes of curves on $Q$ and on Del Pezzo surfaces. $H$ denotes the linear equivalence class of a plane section of $F_d$.

The word class means biliassion class; minimal denotes a curve which is minimal in its biliassion class in the sense of [32]. $M_C$ denotes the Rao module of $C$, and we let

$$\rho(n) = \rho_C(n) = \dim H^1(\mathbb{P}^3, \mathcal{J}_C(n)).$$

When we write $\rho = \{1^2, 2^3\}$, we mean that $\rho(n)$ is zero for $n \neq 1, 2$, while $\rho(1) = 2$ and $\rho(2) = 3$. The sentence $C$ links by $(s, t)$ to $D$ means that $C$ is linked to $D$ by the complete intersection of two surfaces of degrees $s$ and $t$. 
The nullcorrelation bundle $E$ is the rank 2 vector bundle with $c_1 = 0$ and $c_2 = 1$ corresponding to the disjoint union of two lines; “section of the nullcorrelation bundle” means a curve which is the scheme of zeros of a global section $\sigma \in H^0(E(n))$ for some integer $n$.

An extremal curve is a curve of maximal genus among integral nondegenerate curves of a given degree $d$ in $\mathbb{P}^N$: see [1].

The sentence “It does not exist of maximal rank” means there is no integral curve of maximal rank in $\mathbb{P}^3$ with the given spectrum.
### Table 3.1: Spectra of Integral Curves of Degree $\leq 6$

<table>
<thead>
<tr>
<th>$s_{\mathbb{P}^C}$</th>
<th>$d$</th>
<th>$g$</th>
<th>$e$</th>
<th>$s_{\text{max}}$</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>${0}$</td>
<td>1</td>
<td>0</td>
<td>-2</td>
<td>1</td>
<td>Line ACM</td>
</tr>
<tr>
<td>${0, 1}$</td>
<td>2</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>Conic ACM</td>
</tr>
<tr>
<td>${0, 1, 2}$</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>Plane Cubic ACM</td>
</tr>
<tr>
<td>${0, 1^2}$</td>
<td>3</td>
<td>0</td>
<td>-1</td>
<td>2</td>
<td>Twisted cubic ACM</td>
</tr>
<tr>
<td>${0, 1, 2, 3}$</td>
<td>4</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>Plane Quartic ACM</td>
</tr>
<tr>
<td>${0, 1^2, 2}$</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>Elliptic Quartic ACM</td>
</tr>
<tr>
<td>${0, 1^3}$</td>
<td>4</td>
<td>0</td>
<td>-1</td>
<td>2</td>
<td>Rational Quartic Class of 2 disjoint lines.</td>
</tr>
<tr>
<td>${0, 1, 2, 3, 4}$</td>
<td>5</td>
<td>6</td>
<td>2</td>
<td>1</td>
<td>Plane Quintic ACM</td>
</tr>
<tr>
<td>${0, 1^2, 2^2}$</td>
<td>5</td>
<td>2</td>
<td>0</td>
<td>2</td>
<td>Type $(3, 2)$ on Q ACM</td>
</tr>
<tr>
<td>${0, 1^3, 2}$</td>
<td>5</td>
<td>1</td>
<td>0</td>
<td>3</td>
<td>Elliptic Quintic Class of 2 disjoint lines.</td>
</tr>
<tr>
<td>${0, 1^4}$</td>
<td>5</td>
<td>0</td>
<td>-1</td>
<td>3</td>
<td>General Rational Quintic $C_{\mathbb{P}^5}$ dual to $R/(x, y, z^2, zw, w^2)$</td>
</tr>
<tr>
<td>${0, 1, \ldots, 5}$</td>
<td>6</td>
<td>10</td>
<td>3</td>
<td>1</td>
<td>Plane Sextic ACM</td>
</tr>
<tr>
<td>${0, 1^2, 2^2, 3}$</td>
<td>6</td>
<td>4</td>
<td>1</td>
<td>2</td>
<td>Type $(3, 3)$ on Q ACM</td>
</tr>
<tr>
<td>${0, 1^3, 2^3}$</td>
<td>6</td>
<td>3</td>
<td>0</td>
<td>3</td>
<td>Nonspecial of max. rk. ACM</td>
</tr>
<tr>
<td>${0, 1^3, 2^2}$</td>
<td>6</td>
<td>2</td>
<td>0</td>
<td>3</td>
<td>Nonspecial of max. rk. Class of line $\cup$ disjoint conic</td>
</tr>
<tr>
<td>${0, 1^4, 2}$</td>
<td>6</td>
<td>1</td>
<td>0</td>
<td>3</td>
<td>Elliptic Sextic Class of 3 disjoint lines</td>
</tr>
<tr>
<td>${0, 1^5}$</td>
<td>6</td>
<td>0</td>
<td>-1</td>
<td>3</td>
<td>Rational Sextic of max. rk. Minimal</td>
</tr>
</tbody>
</table>
### 3.6. SPECTRA OF INTEGRAL CURVES OF SMALL DEGREE

Table 3.2: Spectra of Integral Curves of Degree 7 and 8.

<table>
<thead>
<tr>
<th>$sp_C$</th>
<th>$d$</th>
<th>$g$</th>
<th>$e$</th>
<th>$s_{max}$</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>{0, 1, \ldots, 6}</td>
<td>7</td>
<td>15</td>
<td>4</td>
<td>1</td>
<td>Plane degree 7.</td>
</tr>
<tr>
<td>{0, 1^2, 2^2, 3^2}</td>
<td>7</td>
<td>6</td>
<td>1</td>
<td>2</td>
<td>Type (4, 3) on Q ACM.</td>
</tr>
<tr>
<td>{0, 1^2, 2^3, 3}</td>
<td>7</td>
<td>5</td>
<td>1</td>
<td>3</td>
<td>$C \sim \text{Line} + 2H$ on $F_3^{\text{smooth}}$ ACM</td>
</tr>
<tr>
<td>{0, 1^2, 2^4}</td>
<td>7</td>
<td>4</td>
<td>0</td>
<td>3</td>
<td>Nonspecial of max. rk. Class of 2 disjoint lines.</td>
</tr>
<tr>
<td>{0, 1^3, 2^3}</td>
<td>7</td>
<td>3</td>
<td>0</td>
<td>3</td>
<td>Nonspecial of max. rk. Links to rational quintic of max. rk.</td>
</tr>
<tr>
<td>{0, 1^4, 2^2}</td>
<td>7</td>
<td>2</td>
<td>0</td>
<td>4</td>
<td>Nonspecial of max. rk. Minimal $\rho = {1^2, 2^3}$</td>
</tr>
<tr>
<td>{0, 1^5, 2}</td>
<td>7</td>
<td>1</td>
<td>0</td>
<td>4</td>
<td>Elliptic max. rk. Minimal $\rho = {1^3, 2^4, 3^1}$</td>
</tr>
<tr>
<td>{0, 1^6}</td>
<td>7</td>
<td>0</td>
<td>-1</td>
<td>4</td>
<td>Rational max. rk. Minimal $\rho = {1^4, 2^5, 3^2}$</td>
</tr>
<tr>
<td>{0, 1, \ldots, 7}</td>
<td>8</td>
<td>21</td>
<td>5</td>
<td>1</td>
<td>Plane degree 8, ACM.</td>
</tr>
<tr>
<td>{0, 1^2, 2^2, 3^2, 4}</td>
<td>8</td>
<td>9</td>
<td>2</td>
<td>2</td>
<td>Type (4, 4) on Q ACM.</td>
</tr>
<tr>
<td>{0, 1^2, 2^2, 3^3}</td>
<td>8</td>
<td>8</td>
<td>1</td>
<td>2</td>
<td>Type (5, 3) on Q. Class of 2 disjoint lines.</td>
</tr>
<tr>
<td>{0, 1^2, 2^3, 3^2}</td>
<td>8</td>
<td>7</td>
<td>1</td>
<td>3</td>
<td>$C \sim \text{Conic} + 2H$ on $F_3^{\text{smooth}}$ ACM</td>
</tr>
<tr>
<td>{0, 1^2, 2^4, 3}</td>
<td>8</td>
<td>6</td>
<td>1</td>
<td>3</td>
<td>$C \sim 2 \text{skew lines} + 2H$ on $F_3^{\text{smooth}}$ Class of 2 disjoint lines.</td>
</tr>
<tr>
<td>{0, 1^3, 2^3, 3}</td>
<td>8</td>
<td>5</td>
<td>1</td>
<td>3</td>
<td>Canonical curve $g = 5$ (B of Max. Rk) Class two disjoint conics</td>
</tr>
<tr>
<td>{0, 1^2, 2^5}</td>
<td>8</td>
<td>5</td>
<td>0</td>
<td>4</td>
<td>Nonspecial max. rk. Minimal in $L(k^2)$</td>
</tr>
<tr>
<td>{0, 1^3, 2^4}</td>
<td>8</td>
<td>4</td>
<td>0</td>
<td>4</td>
<td>Nonspecial of max. rk. Minimal $\rho = {1^1, 2^5, 3^1}$</td>
</tr>
<tr>
<td>{0, 1^4, 2^3}</td>
<td>8</td>
<td>3</td>
<td>0</td>
<td>4</td>
<td>Nonspecial of max. rk. Minimal $\rho = {1^2, 2^4, 3^2}$</td>
</tr>
<tr>
<td>{0, 1^5, 2^2}</td>
<td>8</td>
<td>2</td>
<td>0</td>
<td>4</td>
<td>Nonspecial of max. rk. Minimal $\rho = {1^3, 2^5, 3^1}$</td>
</tr>
<tr>
<td>{0, 1^6, 2}</td>
<td>8</td>
<td>1</td>
<td>0</td>
<td>4</td>
<td>Elliptic max. rk. Minimal $\rho = {1^4, 2^6, 3^4}$</td>
</tr>
<tr>
<td>{0, 1^7}</td>
<td>8</td>
<td>0</td>
<td>-1</td>
<td>4</td>
<td>Rational max. rk. Minimal $\rho = {1^5, 2^7, 3^5}$</td>
</tr>
</tbody>
</table>
### Table 3.3: Spectra of Integral Curves of Degree 9.

<table>
<thead>
<tr>
<th>$s_{PC}$</th>
<th>$g$</th>
<th>$e$</th>
<th>$s_{max}$</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>{0, 1,\ldots, 8}</td>
<td>28</td>
<td>6</td>
<td>1</td>
<td>Plane Curve degree 9 ACM</td>
</tr>
<tr>
<td>{0, 1^2, 2^2, 3^2, 4^2}</td>
<td>12</td>
<td>2</td>
<td>2</td>
<td>Type (5, 4) on Q. ACM</td>
</tr>
<tr>
<td>{0, 1^2, 2^3, 3^2, 4}</td>
<td>10</td>
<td>2</td>
<td>3</td>
<td>Complete Intersection of 2 Cubics. ACM</td>
</tr>
<tr>
<td>{0, 1^2, 2^2, 3^4}</td>
<td>10</td>
<td>1</td>
<td>2</td>
<td>Type (6, 3) on Q. Class 3 disjoint lines</td>
</tr>
<tr>
<td>{0, 1^2, 2^3, 3^4}</td>
<td>9</td>
<td>1</td>
<td>3</td>
<td>$C \sim$ twisted cubic $+ 2H$ on $F_3^{smooth}$ ACM</td>
</tr>
<tr>
<td>{0, 1^2, 2^4, 3^2}</td>
<td>8</td>
<td>1</td>
<td>4</td>
<td>$C \sim$ elliptic quintic $+ H$ on $F_4$ Class of 2 disjoint lines. (cf. [39] Theorem 8.2.7)</td>
</tr>
<tr>
<td>{0, 1^3, 2^4, 3^2}</td>
<td>7</td>
<td>1</td>
<td>3</td>
<td>$\exists$ of maximal rank. $C$ type (6, 3, $\alpha$) with $h(\alpha) = 6$ on $F_4^{\text{ruled}}$. Minimal curve: double line $g = -2$ union a ruling on $F_4^{\text{ruled}}$.</td>
</tr>
<tr>
<td>{0, 1^2, 2^5, 3}</td>
<td>7</td>
<td>1</td>
<td>4</td>
<td>$\exists$ maximal rank class rational quintic max. rank. cf. [39] Example 5.4.9 p.70</td>
</tr>
<tr>
<td>{0, 1^3, 2^2, 3^4}</td>
<td>6</td>
<td>1</td>
<td>4</td>
<td>Type (6, 2, 2, 2, 2, 1) on Del Pezzo quartic Class rational quartic union disjoint line</td>
</tr>
<tr>
<td>{0, 1^2, 2^6}</td>
<td>6</td>
<td>0</td>
<td>4</td>
<td>Nonspecial of max. rk. Minimal $\rho = {2^3, 3^2}$</td>
</tr>
<tr>
<td>{0, 1^3, 2^5}</td>
<td>5</td>
<td>0</td>
<td>4</td>
<td>Nonspecial of max. rk. Minimal $\rho = {1^3, 2^4, 3^3}$</td>
</tr>
<tr>
<td>{0, 1^4, 2^4}</td>
<td>4</td>
<td>0</td>
<td>4</td>
<td>Nonspecial of max. rk. Minimal $\rho = {1^4, 2^5, 3^4}$</td>
</tr>
<tr>
<td>{0, 1^5, 2^3}</td>
<td>3</td>
<td>0</td>
<td>4</td>
<td>Nonspecial of max. rk. Minimal $\rho = {1^5, 2^6, 3^5}$</td>
</tr>
<tr>
<td>{0, 1^6, 2^2}</td>
<td>2</td>
<td>0</td>
<td>5</td>
<td>Nonspecial of max. rk. Minimal $\rho = {1^4, 2^7, 3^6}$</td>
</tr>
<tr>
<td>{0, 1^7, 2}</td>
<td>1</td>
<td>0</td>
<td>5</td>
<td>Elliptic max. rk. Minimal $\rho = {1^5, 2^8, 3^7, 4^1}$</td>
</tr>
<tr>
<td>{0, 1^8}</td>
<td>0</td>
<td>-1</td>
<td>5</td>
<td>Rational max. rk. Minimal $\rho = {1^6, 2^9, 3^8, 4^2}$</td>
</tr>
</tbody>
</table>
3.6. SPECTRA OF INTEGRAL CURVES OF SMALL DEGREE

Table 3.4: Spectra of Integral Curves of Degree 10.

<table>
<thead>
<tr>
<th>$sp_C$</th>
<th>$g$</th>
<th>$e$</th>
<th>$s_{max}$</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>{0, 1, ..., 9}</td>
<td>36</td>
<td>7</td>
<td>1</td>
<td>Plane Curve degree 10</td>
</tr>
<tr>
<td>{0, 1^2, 2^2, 3^2, 4^2, 5}</td>
<td>16</td>
<td>3</td>
<td>2</td>
<td>Type (5, 5) on Q.</td>
</tr>
<tr>
<td>{0, 1^2, 2^2, 3^2, 4^3}</td>
<td>15</td>
<td>2</td>
<td>2</td>
<td>Type (6, 4) on Q.</td>
</tr>
<tr>
<td>{0, 1^2, 2^3, 3^3, 4}</td>
<td>12</td>
<td>2</td>
<td>3</td>
<td>$C \sim E_6 + 3H$ on $F_3^{smooth}$.</td>
</tr>
<tr>
<td>{0, 1^2, 2^2, 3^2}</td>
<td>12</td>
<td>1</td>
<td>2</td>
<td>Type (7, 3) on Q.</td>
</tr>
<tr>
<td>{0, 1^2, 2^4, 3^2, 4}</td>
<td>11</td>
<td>2</td>
<td>4</td>
<td>Section nullcorrelation bundle.</td>
</tr>
<tr>
<td>{0, 1^2, 2^3, 3}</td>
<td>11</td>
<td>1</td>
<td>4</td>
<td>$\exists$ by (3.1.3).</td>
</tr>
<tr>
<td>{0, 1^2, 2^4, 3}</td>
<td>10</td>
<td>1</td>
<td>4</td>
<td>$\exists$ by [39], 5.3.4.</td>
</tr>
<tr>
<td>{0, 1^3, 2^3, 3}</td>
<td>9</td>
<td>1</td>
<td>3</td>
<td>It does not exist of max. rk.</td>
</tr>
<tr>
<td>{0, 1^2, 2^5, 3^2}</td>
<td>9</td>
<td>1</td>
<td>4</td>
<td>$C \sim$ elliptic sextic $+ H$ on $F_3$;</td>
</tr>
<tr>
<td>{0, 1^3, 2^4, 3^2}</td>
<td>8</td>
<td>1</td>
<td>4</td>
<td>$C \sim$ conic + $2H \sim (7, 3, 2, 2, 2)$ on a Del Pezzo quartic.</td>
</tr>
<tr>
<td>{0, 1^2, 2^6, 3}</td>
<td>8</td>
<td>1</td>
<td>4</td>
<td>$\exists$ of max. rk. by [39], 5.3.4.</td>
</tr>
<tr>
<td>{0, 1^3, 2^5, 3}</td>
<td>7</td>
<td>1</td>
<td>4</td>
<td>$C \sim E_4 + E_5 + 2H \sim (6, 2, 2, 2, 1, 1)$ on a Del Pezzo quartic.</td>
</tr>
<tr>
<td>{0, 1^2, 2^7}</td>
<td>7</td>
<td>0</td>
<td>4</td>
<td>Nonspecial of max. rk.</td>
</tr>
<tr>
<td>{0, 1^4, 2^4, 3}</td>
<td>6</td>
<td>1</td>
<td>4</td>
<td>It does not exist of max. rk.</td>
</tr>
<tr>
<td>{0, 1^9-9, 2^9}</td>
<td>g</td>
<td>0</td>
<td>5</td>
<td>Nonspecial of max. rk.</td>
</tr>
<tr>
<td>{0, 1^9}</td>
<td>0</td>
<td>-1</td>
<td>5</td>
<td>Rational of max. rk.</td>
</tr>
</tbody>
</table>

Note: $g$ denotes the genus, $e$ the number of exceptional divisors, and $s_{max}$ the maximum spectral number.
### Table 3.5: Spectra of Integral Curves of Degree 11 with $g \geq 11$

<table>
<thead>
<tr>
<th>$\text{sPC}$</th>
<th>$g$</th>
<th>$e$</th>
<th>$s_{\text{max}}$</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>${0, 1, \ldots, 10}$</td>
<td>45</td>
<td>8</td>
<td>1</td>
<td>Plane Curve degree 11 ACM</td>
</tr>
<tr>
<td>${0, 1^2, 2^2, 3^2, 4^2, 5^2}$</td>
<td>20</td>
<td>3</td>
<td>2</td>
<td>Type $(6,5)$ on $Q$ ACM</td>
</tr>
<tr>
<td>${0, 1^2, 2^2, 3^2, 4^4}$</td>
<td>18</td>
<td>2</td>
<td>2</td>
<td>Type $(7,4)$ on $Q$ Class three disjoint lines.</td>
</tr>
<tr>
<td>${0, 1^2, 2^4, 3^4, 4^2}$</td>
<td>15</td>
<td>2</td>
<td>3</td>
<td>$\exists$ by Theorem 3.1.3 ACM.</td>
</tr>
<tr>
<td>${0, 1^2, 2^4, 3^4, 4}$</td>
<td>14</td>
<td>1</td>
<td>4</td>
<td>$\exists$ by Theorem 3.1.3 ACM.</td>
</tr>
<tr>
<td>${0, 1^2, 2^2, 3^6}$</td>
<td>14</td>
<td>2</td>
<td>4</td>
<td>Type $(8,3)$ on $Q$ Class 5 disjoint lines on $Q$.</td>
</tr>
<tr>
<td>${0, 1^2, 2^4, 3^3, 4}$</td>
<td>13</td>
<td>2</td>
<td>4</td>
<td>$\exists$ of max. rk. by [39], 5.3.4 : $C \sim D + H$ on $F_5$, where $D$ is nonspecial of max. rk. with $d = 6$ and $g = 2$. Class of line union disjoint conic.</td>
</tr>
<tr>
<td>${0, 1^2, 2^4, 3^3}$</td>
<td>13</td>
<td>1</td>
<td>4</td>
<td>$\exists$ of max. rk. by [39], 5.3.4 : $C \sim D + H$ on $F_4$, where $D$ is nonspecial of max. rk. with $d = 7$ and $g = 4$. Class of 2 disjoint lines.</td>
</tr>
<tr>
<td>${0, 1^3, 2^3, 3^3, 4}$</td>
<td>12</td>
<td>2</td>
<td>3</td>
<td>Type $(7,3, \alpha)$ with $h(\alpha) = 7$ on $F_3^{\text{ruled}}$. Minimal curve is a double line with $g = -3$.</td>
</tr>
<tr>
<td>${0, 1^2, 2^5, 3^2, 4}$</td>
<td>12</td>
<td>2</td>
<td>4</td>
<td>$\exists$ of max. rk. by [39], 5.3.4 : $C \sim D + H$ on $F_5$, where $D$ is elliptic of max. rk. with $d = 6$. $C$ is subcanonical in class of 3 disjoint lines.</td>
</tr>
<tr>
<td>${0, 1^2, 2^4, 3^4}$</td>
<td>12</td>
<td>1</td>
<td>4</td>
<td>$\exists$ of max. rk. by [39], 5.3.4 : $C \sim D + H$ on $F_4$, where $D$ is nonspecial of max. rk. with $d = 7$ and $g = 3$. $C$ links by $(4,4)$ to rational quintic of max. rk.</td>
</tr>
<tr>
<td>${0, 1^3, 2^4, 3^4}$</td>
<td>11</td>
<td>1</td>
<td>3</td>
<td>Type $(8,5, \alpha)$ with $h(\alpha) = 8$ on $F_3^{\text{ruled}}$. Minimal curve is a double line with $g = -2$ union 3 rulings.</td>
</tr>
<tr>
<td>${0, 1^2, 2^5, 3^3}$</td>
<td>11</td>
<td>1</td>
<td>4</td>
<td>$\exists$ of max. rk. by [39], 5.3.4 : $C \sim D + H$ on $F_4$, where $D$ is nonspecial of max. rk. with $d = 7$ and $g = 2$. $D$ is minimal with $\rho = {1^2, 2^5}$.</td>
</tr>
</tbody>
</table>
3.6. SPECTRA OF INTEGRAL CURVES OF SMALL DEGREE

Table 3.6: Spectra of Integral Curves of Degree 11 with $g \leq 10$

<table>
<thead>
<tr>
<th>$sp_C$</th>
<th>$g$</th>
<th>$e$</th>
<th>$s_{max}$</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>${0,1^3,2^4,3^3}$</td>
<td>10</td>
<td>1</td>
<td>4</td>
<td>It doesn’t exists of maximal rank ($s(C) = 5$): in $\mathbb{P}^4$ lies on 2 independent quadrics. $\exists$ with $s(C) = 4$ on Del Pezio quartic: $C \sim (7,2,2,2,2,2)$.</td>
</tr>
<tr>
<td>${0,1^2,2^6,3^2}$</td>
<td>10</td>
<td>1</td>
<td>5</td>
<td>$\exists$ of maximal rank by [28] or [33]: minimal curve general Rao module with $\rho = {2^4,3^4}$.</td>
</tr>
<tr>
<td>${0,1^3,2^5,3^2}$</td>
<td>9</td>
<td>1</td>
<td>5?</td>
<td>$\exists$ with $s(C) = 4$ on Del Pezio quartic has $s = 4$.</td>
</tr>
<tr>
<td>${0,1^2,2^7,3}$</td>
<td>9</td>
<td>1</td>
<td>5</td>
<td>$\exists$ of maximal rank by [11] or [49]: Minimal.</td>
</tr>
<tr>
<td>${0,1^4,2^4,3^2}$</td>
<td>8</td>
<td>1</td>
<td>4</td>
<td>It does not exist of max. rk. by Proposition 3.4.9. $\exists$ with $s(C) = 4$, projection of extremal $C$ on quartic rational scroll in $\mathbb{P}^5$.</td>
</tr>
<tr>
<td>${0,1^3,2^6,3}$</td>
<td>8</td>
<td>1</td>
<td>5?</td>
<td>$\exists$ on Del Pezio quintic: $C \sim (6,2,2,1,1) \sim E_4 + 2H$.</td>
</tr>
<tr>
<td>${0,1^2,2^8}$</td>
<td>8</td>
<td>0</td>
<td>5</td>
<td>$\exists$ on Del Pezio quintic: $C \sim (6,2,2,1,1) \sim E_4 + 2H$.</td>
</tr>
<tr>
<td>${0,1^4,2^5,3}$</td>
<td>7</td>
<td>1</td>
<td>5?</td>
<td>$\exists$ on Del Pezio quintic: $C \sim (6,2,2,1,1) \sim E_4 + 2H$.</td>
</tr>
<tr>
<td>${0,1^3,2^7}$</td>
<td>7</td>
<td>0</td>
<td>5</td>
<td>$\exists$ on Del Pezio quintic: $C \sim (6,2,2,1,1) \sim E_4 + 2H$.</td>
</tr>
<tr>
<td>${0,1^4,2^6}$</td>
<td>6</td>
<td>0</td>
<td>5</td>
<td>$\exists$ on Del Pezio quintic: $C \sim (6,2,2,1,1) \sim E_4 + 2H$.</td>
</tr>
<tr>
<td>${0,1^5,2^5}$</td>
<td>5</td>
<td>0</td>
<td>5</td>
<td>$\exists$ on Del Pezio quintic: $C \sim (6,2,2,1,1) \sim E_4 + 2H$.</td>
</tr>
<tr>
<td>${0,1^6,2^4}$</td>
<td>4</td>
<td>0</td>
<td>5</td>
<td>$\exists$ on Del Pezio quintic: $C \sim (6,2,2,1,1) \sim E_4 + 2H$.</td>
</tr>
<tr>
<td>${0,1^7,2^3}$</td>
<td>3</td>
<td>0</td>
<td>5</td>
<td>$\exists$ on Del Pezio quintic: $C \sim (6,2,2,1,1) \sim E_4 + 2H$.</td>
</tr>
<tr>
<td>${0,1^8,2^2}$</td>
<td>2</td>
<td>0</td>
<td>5</td>
<td>$\exists$ on Del Pezio quintic: $C \sim (6,2,2,1,1) \sim E_4 + 2H$.</td>
</tr>
<tr>
<td>${0,1^9,2^1}$</td>
<td>1</td>
<td>0</td>
<td>5</td>
<td>$\exists$ on Del Pezio quintic: $C \sim (6,2,2,1,1) \sim E_4 + 2H$.</td>
</tr>
<tr>
<td>${0,1^{10}}$</td>
<td>0</td>
<td>-1</td>
<td>6</td>
<td>$\exists$ on Del Pezio quintic: $C \sim (6,2,2,1,1) \sim E_4 + 2H$.</td>
</tr>
</tbody>
</table>
3.7 A few Remarks on the Tables

We would like to make a couple of remarks on Table 3.4. To construct a curve with spectrum \( \{0, 1^3, 2^4, 3^2\} \) and \( s(C) = 4 \), which is \( s_{\text{max}} \) for this spectrum, a natural thing to do would be to start with a nice curve with spectrum \( \{0^2, 1^3, 2\} \), as for example a line disjoint union an elliptic quintic, and then bilink up on a surface of degree four. Or we could look at the dual biliaison class, and start with a smooth rational curve of degree six on a smooth cubic and try to link it by the complete intersection of two quartics to get our desired \( C \). The trouble with this approach is that we don’t know how to guarantee that we end up with a smooth \( C \): it seems to us that the Bertini’s type Theorems in [39] would not apply in this situation. That is why we look at the Del Pezzo quartic surface in \( \mathbb{P}^4 \) to construct \( C \). Then we project to \( \mathbb{P}^3 \); the resulting curve is automatically contained in an integral quartic in \( \mathbb{P}^3 \), and an easy argument shows that it cannot lie on a cubic surface. A posteriori, we see that such a \( C \) actually is in the biliaison class of a curve with spectrum \( \{0^2, 1^3, 2\} \).

Another interesting spectrum is \( \{0, 1^4, 2^4, 3\} \). It is easy to see that any integral curve with this spectrum is canonical, that is, \( \omega_C \cong \mathcal{O}_C(1) \) (for example, the corresponding reflexive sheaf \( \mathcal{F} \) is in fact locally free because \( c_3(\mathcal{F}) = 0 \)). By Proposition 3.4.9 there is no such curve of maximal rank in \( \mathbb{P}^3 \).

3.8 The Spectrum of an Integral Curve is Not Necessarily the Spectrum of a Smooth Curve

Before we proceed, we give an example of a set of integers with multiplicities which occurs as the spectrum of an integral curve, but not as the spectrum of a smooth curve.

Hartshorne [23] gave an example of an integral curve \( C \) of degree 14 and arithmetic genus 23 which is not smoothable. We show now that the spectrum of \( C \) is not the spectrum of any smooth curve (this of course does not imply that \( C \) is not smoothable). The curve \( C \) is constructed in \( \mathbb{P}^3 \) on the ruled cubic surface of section 3.5 where it is linearly equivalent to a double line with \( g = -3 \) plus four times the plane section: \( C \) has type \( (9, 4, \alpha) \) with \( h(\alpha) = 7 \). The double line has spectrum \( \{-2, 0\} \), hence by Proposition 2.2.1 the spectrum of \( C \) is

\[ \{0, 1^2, 2^4, 3^3, 4^3, 5\}. \]
3.8. THE SPECTRUM OF AN INTEGRAL CURVE IS NOT NECESSARILY THE SPECTRUM OF A SMOOTH CURVE

Such a curve cannot be smooth: the ruled cubic is the projection of a smooth cubic scroll in \( \mathbb{P}^4 \) (cf. Section 3.5); if \( C \) were smooth, this projection would give an isomorphism between \( C \) and its preimage in \( \mathbb{P}^4 \) and their spectra would be the same; in particular, \( C \) could not be linearly normal in \( \mathbb{P}^3 \), while its spectrum gives \( h^0 \mathcal{O}_C(1) = 3 \). There is such a \( C \) integral with only one node, and it is the projection of a curve on the scroll of genus 22 with spectrum \( \{0, 1^3, 2^3, 3^3, 4^3, 5\} \).

Now we claim that any integral curve \( D \) with spectrum

\[ \{0, 1^2, 2^4, 3^3, 4^3, 5\} \]

lies on a cubic surface with a double line and it is not smooth. First of all, the spectrum gives \( h^0 \mathcal{O}_D(4) = 34 \), hence \( D \) is contained on some surface of degree 4, and \( D \) cannot be contained in a plane or in an integral quadric surface because of the explicit calculations of the spectra of curves on these surfaces (the latter statement is also a consequence of the biliaison trick).

Therefore \( s(C) \) is either 3 or 4, and we claim it is 3: if \( D \) were not contained on a cubic surface, then \( D \) would lie on an integral quartic, and by the biliaison trick, on this quartic \( D - 3H \) would be linearly equivalent to an effective curve with spectrum \( \{-1, 1\} \), which is absurd. Thus \( C \) is contained on a cubic surface \( F \), necessarily integral. On \( F \), by the biliaison trick, \( D - 4H \) is linearly equivalent to an effective curve \( B \) with spectrum \( \{-2, 0\} \). The only curve with such a spectrum is a double structure on a line \( L \) with \( g = -3 \); if \( F \) were generically smooth along \( L \), any double structure on \( L \) contained in \( F \) would have \( g \geq -2 \). Hence \( L \) is a double line for \( F \), so \( F \) is either a ruled cubic surface or a cone over a singular irreducible plane cubic. At this point, we could probably check by hand (as we did above for the general ruled cubic) that there is no smooth curve with our spectrum on these surfaces. Instead, we apply [39], Proposition 8.2.4: in our case, we have \( A = 3 \), \( h_D = 4 \), \( t(D) \geq 6 \) and \( \theta_0 = 6 \), and Nollet’s result implies that there is no smooth curve with these invariants in the biliaison class of a double line with arithmetic genus \(-3\).

We think the degree 13 spectrum

\[ sp = \{0, 1^2, 2^4, 3^3, 4^3\} \]

is another example of a spectrum which is possible for an integral curve, but not for a smooth curve. The situation in this case is slightly more complicated. Arguing as above, we see that any integral curve with spectrum \( sp \) lies on a cubic surface \( F \) in \( \mathbb{P}^3 \), and on \( F \) any such curve bilinks down to a curve with spectrum \( \{-1, 0, 1^2\} \). This is not the spectrum of any curve on the smooth cubic surface, so that \( F \) must be a singular integral cubic surface. There are integral curves on the ruled cubic surface with spectrum
$sp$: they have type either $(8, 3, \alpha)$ with $h(\alpha) = 6$, or $(9, 5, \alpha)$ with $h(\alpha) = 7$, and none of them is smooth.
Bibliography


