FOOTNOTE TO A PAPER BY HARTSHORNE

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Dedicated to Robin Hartshorne on occasion of his sixtieth birthday.

Abstract. We prove that every curve in the biliaison class of an extremal curve in \( \mathbb{P}^3 \) can be connected to an extremal curve.

0. Introduction

Hartshorne has recently shown [2, 3.4-3.5] that every curve in the biliaison classes of a line and of two skew lines in \( \mathbb{P}^3 \) can be connected to an extremal curve. In this note we show how to extend his result to curves in the biliaison class of an arbitrary extremal curve.

Let us explain these statements. We use the notation and terminology of [2]. In particular, we denote by \( H_{d,g} \) the Hilbert scheme of locally Cohen-Macaulay curves of degree \( d \) and arithmetic genus \( g \) in \( \mathbb{P}^3 \), the projective 3-space over an algebraically closed field \( k \). A curve \( E \) is extremal if

\[
(1) \quad h^1(\mathbb{P}^3, \mathcal{I}_{C,\mathbb{P}^3}(n)) \leq h^1(\mathbb{P}^3, \mathcal{I}_{E,\mathbb{P}^3}(n))
\]

for all integers \( n \), and for all locally Cohen-Macaulay curves \( C \) of degree \( d = \deg E \) and genus \( g = g(E) \) (at least in characteristic zero the analogous inequalities for \( h^0 \) and \( h^2 \) also hold true).

It can be shown [3, 9.5] that a curve of degree \( d \) is extremal if and only if it can be specialized with constant cohomology to a curve containing a planar subcurve of degree \( d-1 \). Extremal curves have been introduced and studied by Martin-Deschamps and Perrin [6],[7] (their definition of an extremal curve is slightly more restrictive than the one we borrowed from [2]). They have shown that, whenever \( H_{d,g} \) is nonempty, there exist extremal curves of degree \( d \) and genus \( g \), and in fact they form an irreducible component \( E_{d,g} \) of \( H_{d,g} \).

We say that a curve \( C \) of degree \( d \) and genus \( g \) can be connected to an extremal curve, if \( C \) lies in the connected component of \( H_{d,g} \) that contains \( E_{d,g} \). Note that semicontinuity does not preclude the possibility that every curve can be connected to an extremal curve, that is, it is conceivable that \( H_{d,g} \) is connected, and no counterexample is known. The state of the art regarding this problem is very well described in [2], where a number of results are proven. In particular, Hartshorne shows that every curve in the biliaison classes of a line and of two skew lines can be connected to an extremal curve. In this paper, we show more generally that a curve in the biliaison class of an arbitrary extremal curve can be connected to an extremal curve. Our result is based on the techniques of [2], the only new idea being the construction of an irreducible family of multiple lines that contain a fixed multiple line, which is explained in section 1 and has its origin in the paper [1]. This construction allows us to give a positive answer to the question raised in [2, 4.3].

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I am most grateful to Professor Hartshorne for providing a nice general problem, useful techniques to attack it, and many interesting questions. I am also indebted to him for suggestions that substantially improved this note.

Added in proof Independently, D. Perrin [8] has obtained the main result of this paper by an entirely different method. I would like to thank him for letting me publish this note.

1. SMALL EXTENSIONS OF MULTIPLE CURVES

We work over a fixed algebraically closed field $k$. We define a curve to be a one dimensional scheme of finite type over $k$ with no zero dimensional associated points. A locally Cohen-Macaulay multiplicity structure $Z$ on a reduced irreducible curve $Y$ is a curve $Z$ such that $Z_{\text{red}} = Y$.

**Lemma 1.1.** Suppose $W$ is a locally Cohen-Macaulay multiplicity structure on the smooth irreducible curve $Y$. Let $Z \subset W$ be a closed subscheme containing $Y$, and suppose $\mult_Y(W) = \mult_Y(Z) + 1$.

Then $\mathcal{I}_{Z,W}$ is annihilated by $\mathcal{I}_{Y,W}$, and, as an $\mathcal{O}_Y$-module, it is locally free of rank one.

**Proof.** Let $T$ be the subscheme of $W$ defined by the ideal sheaf $\mathcal{I}_{Z,W}\mathcal{I}_{Y,W}$: $T$ contains $Z$, and by Nakayama’s lemma $\mult_Y(T) > \mult_Y(Z) + 1$. Thus $\mult_Y(T) = \mult_Y(W)$, which implies that $T = W$ since $W$ is locally Cohen-Macaulay. This shows $\mathcal{I}_{Z,W}$ is killed by $\mathcal{I}_{Y,W}$, hence we may consider it an $\mathcal{O}_Y$-module. It is locally free because it is a subsheaf of $\mathcal{O}_W$, which has depth one at closed points, and its rank is the difference between the multiplicities of $W$ and $Z$ along $Y$. □

**Definition 1.2.** Given a locally Cohen-Macaulay multiplicity structure $Z$ on the smooth irreducible curve $Y$, a small extension of $Z$ is a locally Cohen-Macaulay multiplicity structure $W$ on $Y$ that contains $Z$ and such that $\mathcal{I}_{Z,W}$ is annihilated by $\mathcal{I}_{Y,W}$. Then $\mathcal{I}_{Z,W}$ may be considered an $\mathcal{O}_Y$-module, and as such is isomorphic to a coherent locally free sheaf $\mathcal{G}$ on $Y$.

We say that $W$ is a small extension of type $\mathcal{G}$.

**Example 1.3.** By the previous lemma, if $\mult_Y(W) = \mult_Y(Z) + 1$, $W$ is a small extension of $Z$ and its type $\mathcal{G}$ is an invertible sheaf.

For the rest of this section we work in an ambient scheme $X$, which we assume to be of finite type over $k$. When we write $T \subset X$, we mean $T$ is a closed subscheme of $X$, and in this case we denote by $\mathcal{I}_T$ the ideal sheaf of $T$ in $X$. Given a locally Cohen-Macaulay multiplicity structure $Z \subset X$ on the smooth irreducible curve $Y$, we would like to parametrize the set of small extensions of $Z$ in $X$ of a given type $\mathcal{G}$.

We define a locally free sheaf $\mathcal{F}_{Z,X}$ on $Y$ as follows: consider the conormal sheaf $\mathcal{I}_Z/\mathcal{I}_Z^2$ of $Z$ in $X$, restrict it to $Y$, and then quotient out the torsion subsheaf to obtain $\mathcal{F}_{Z,X}$.

**Proposition 1.4.** Let $X$ be a scheme of finite type over $k$ containing the smooth irreducible curve $Y$. Let $Z \subset X$ be a locally Cohen-Macaulay multiplicity structure on $Y$, and fix a coherent locally free sheaf $\mathcal{G}$ on $Y$. Let $V$ denote the open subset of $\text{Hom}_Y(\mathcal{F}_{Z,X}, \mathcal{G})$ consisting of surjective morphisms. The group $G$ of $\mathcal{O}_Y$-automorphisms of $\mathcal{G}$ acts (by composition) on $V$, and we let $V/G$ denote the set theoretic quotient.

There is a one to one correspondence between $V/G$ and the set of small extensions of $Z$ in $X$ of type $\mathcal{G}$. The multiplicity structure $W_\phi$ corresponding to $\phi \in V$ is defined as follows: if $\mathcal{F}_{Z,X} = \mathcal{I}_Z/\mathcal{J}$, then $\mathcal{I}_{W_\phi}/\mathcal{J} = \text{Ker} \phi$. 
Proof. Given an epimorphism $\phi : F_{Z,X}^Y \to G$, we define $W_\phi$ by letting $I_{W_\phi}/J = Ker\phi$. Then $I_{Z,W} \cong G$, so that we have an exact sequence

$$0 \to G \to \mathcal{O}_W \to \mathcal{O}_Z \to 0.$$  

This sequence shows that $W$ has no embedded points, and is a small extension of $Z$ of type $G$. On the other hand, if $W$ is a small extension of $Z$ in $X$ of type $G$, the epimorphism $I_Z \to I_{Z,W} \cong G$ factors through $I_Z \otimes \mathcal{O}_Y$, because $G$ is an $\mathcal{O}_Y$-module, and then through $F_{Z,X}^Y$, because $G$ is torsion free. Hence it induces an epimorphism $\phi : F_{Z,X}^Y \to G$, and clearly $W = W_\phi$.

Finally, $W_\phi = W_\psi$ if and only if $\phi$ and $\psi$ have the same kernel, that is, there is an automorphism $\sigma$ of $G$ such that $\phi = \sigma \circ \psi$. \qed

**Corollary 1.5.** Let $X$ be a projective scheme over $k$, and let $Y \subset X$ be a smooth irreducible curve. Fix a locally free coherent sheaf $G$ on $Y$, and a locally Cohen-Macaulay multiplicity structure $Z \subset X$ on $Y$. Suppose the set of small extensions of $Z$ in $X$ of type $G$ is nonempty. Then there exists an irreducible constructible subspace of the topological space underlying the Hilbert scheme $Hilb(X)$, whose closed points parametrize small extensions of $Z$ in $X$ of type $G$.

**Proof.** Let $\mathcal{F} = F_{Z,X}^Y$, and let $T = \text{Spec } k[V^*]$ be the affine space associated to the $k$-vector space $V = \text{Hom}_{\mathcal{O}_Y}(\mathcal{F}, G)$. Then a closed point $t \in T$ corresponds to a morphism $\phi_t \in V$. If $q : Y \times T \to Y$ is the first projection, there is a universal morphism, $\phi : q^*\mathcal{F} \to q^*G$ on $Y \times T$ with the following property: for each closed point $t \in T$, the morphism induced by $\psi$ on the fibre over $t$ is the morphism $\phi_t \in V$: cf. [4, 3.1].

Now let $U$ be the open subscheme of the $T$ which corresponds to surjective morphisms. $U$ is nonempty because of the assumption on the existence of some extension of $Z$ in $X$ of type $G$. Let $p_X : X \times U \to X$ be the first projection. $\phi$ induces a morphism $\psi : p_X^*\mathcal{I}_Z \to p_X^*G$. The kernel of $\psi$ is the ideal sheaf of a closed subscheme $W \subset X \times U$. The fibre of the projection map $W \to U$ over a closed point $t \in U$ is the small extension $W_{\phi_t}$ constructed above. In particular, all fibres over closed points have the same Hilbert polynomial. Since $U$ is smooth and irreducible, $W$ is flat over $U$, hence it gives rise to a morphism $f : U \to Hilb(X)$. By construction the closed points of $f(U)$ correspond to the closed subschemes $W_{\phi_t}$ of $X$, hence by the previous proposition to the small extensions of $Z$ in $X$ of type $G$. Note that, since $f$ is a morphism of finite type, $f(U)$ is constructible. \qed

## 2. Biliaison

In this section we work with curves in $\mathbb{P}^3$. Our main result is that every curve in the biliaison class of an extremal curve can be connected to an extremal curve. This follows from [2], once we have established Lemma 2.1 below.

For any curve $C$, let $s_0(C)$ denote the least degree of a surface containing $C$. For any $s \geq s_0(C)$ and any $h > 0$, $C(s,h)$ denotes a curve obtained from $C$ by an elementary biliaison of height $h$ on a surface of degree $s$ [5, III 2].

Hartshorne [2, 3.2] has shown that, if $E$ is a nonplanar extremal curve and $2 \leq s \leq \deg E$, then the biliaison $E(s,1)$ can be connected to an extremal curve on surfaces of degree 2 if $s = 2$ or 3 if $s \geq 3$. This is also true for $s > \deg E$ by the following lemma, which gives an answer to the question raised in [2, 4.3]. Independently, Perrin[8] has obtained the same result by an entirely different method.
Lemma 2.1. If $E$ is a nonplanar extremal curve, for any $s > \deg E$, the biliaison $E(s, 1)$ can be connected to an extremal curve by curves lying in surfaces of degree $3 = s_0(E(s, 1))$.

Proof. Let $d$ be the degree of $E$, and fix $s > d$. Note that $s_0(E) = 2$ because $E$ is extremal and nonplanar. Since $s > d \geq 2$, it follows that $s_0(E(s, 1)) = 3$.

Any extremal curve of degree $d \geq 2$ can be deformed with constant cohomology to an extremal curve $E'$ that is supported on a line $L$ and contains a planar multiple line $P$ of degree $d - 1$. We may replace $E$ with $E'$. We fix a plane $H$ through $P$ of equation $x = 0$. If $K$ is a plane of equation $y = 0$ that cuts out the line $L$ on $H$, $P$ will have equations $x = y^{d-1} = 0$. Then $E$ is contained in the multiple plane $dK$, hence in $sK$ because $s > d$. Therefore we may consider the curve $W = E(s, 1)$ obtained as the divisor $E + sK.H$ on the surface $sK$.

$W$ is supported on the line $L$, so it is a locally Cohen-Macaulay multiplicity structure on $L$ of degree $s + d$. Furthermore, the divisor $Z = P + sK.H$ on the surface $sK$ is contained in $W$, and is a multiple line of degree $s + d - 1$. It follows from 1.1 that $I_{Z,W} \cong O_L(m)$ for some integer $m$. We can compute $m$ in terms of the arithmetic genera of $P$ and $E$:

$$m = p_a(Z) - p_a(W) - 1 = p_a(P) - p_a(E) - 2 \geq -d.$$ 

The homogeneous ideal of $Z$ is generated by $x^2$, $xy^{d-1}$ and $y^s$, therefore the conormal sheaf of $Z$ restricted to $L$ is locally free, and splits as

$$\mathcal{F}_{Z,\mathbb{P}^3}^L \cong O_L(-2) \oplus O_L(-d) \oplus O_L(-s).$$

By corollary 1.5, $W$ varies in the irreducible family $C$ of small extensions of $Z$ of type $O_L(m)$. Since $m \geq -d$, we can find a surjective morphism

$$O_L(-2) \oplus O_L(-d) \oplus O_L(-s) \to O_L(m)$$

which vanishes on the first summand $O_L(-2)$. It follows that we can find an extension $W_0 \in C$ which lies in a double plane. Note that, since $Z$ lies in a surface of degree two, all curves in the family $C$ lie in a cubic surface. By [3, 8.1], $W_0$ is connected to an extremal curve by a family of curves lying in a double plane, thus $W = E(s, 1)$ is connected to an extremal curve by a family of curves lying in cubic surfaces.

We can now generalize [2, 3.3]:

Proposition 2.2. Let $C$ be connected to an extremal curve $E$ by flat families of curves in surfaces of degree $s_1$. Then for any $s \geq s_1$, the biliaison $C(s, 1)$ can be connected to an extremal curve $E'$ by flat families of curves in surfaces of degree $s_1$ if $s = s_1$, or $s_1 + 1$ if $s > s_1$.

Proof. If $C_i$ is a flat family of curves lying in surfaces of degree $s_1$, then for any $s \geq s_1$ one can form a flat family $C_i(s, 1)$ whose fibres are the elementary biliaisons of the curves $C_i$. The curves $C_i(s, 1)$ lie on surfaces of degree $s_1$ if $s = s_1$, or $s_1 + 1$ if $s > s_1$. Therefore we can connect $C(s, 1)$ to $E(s, 1)$ by flat families of curves in surfaces of the desired degree. By [2, 3.1-3.2] and Lemma 2.1 we can then connect $E(s, 1)$ to an extremal curve $E'$ by flat families of curves in surfaces of the desired degree.

If $C_0$ and $C$ are two curves in the same biliaison class, $C$ dominates $C_0$ if there is a sequence of curves $C_1, \ldots, C_h$ such that for each $i = 0, \ldots, h - 1$, $C_{i+1} = C_i(s_i, 1)$, and $C_h$ is a deformation of $C$ with constant cohomology and Rao module (by this we mean that $C_h$ and $C$ belong to an irreducible flat family of curves, all with the same cohomology and Rao
module). The Lazarsfeld-Rao property [5] says that, if \( C_0 \) is a minimal curve in a biliaison class \( C \), every curve in \( C \) dominates \( C_0 \).

**Theorem 2.3.** Suppose \( C_0 \) can be connected to an extremal curve by curves lying in surfaces of degree \( s_0(C_0) \). Then any curve \( C \) that dominates \( C_0 \) in the biliaison class of \( C_0 \) can be connected to an extremal curve by curves lying in surfaces of degree \( s_0(C) \).

In particular, if \( C_0 \) is minimal, every curve \( C \) in the biliaison class of \( C_0 \) can be connected to an extremal curve by curves lying in surfaces of degree \( s_0(C) \).

**Proof.** By Proposition 2.2, for any \( s \geq s_0(C_0) \) the biliaison \( C_1 = C_0(s, 1) \) can be connected to an extremal curve by flat families of curves in surfaces of degree \( s_0(C_1) \). Thus we can iterate the argument and conclude.

**Remark 2.4.** It is not likely that an arbitrary minimal curve satisfies the condition of the theorem. The first example that comes to mind is \( C_0 \) equal to the disjoint union of four lines on a quadric; see the discussion in section 4 of [2].

The following corollary generalizes [2, 3.4-3.5-3.7].

**Corollary 2.5.** Every curve \( C \) in the biliaison class of an extremal curve \( E \) can be connected to an extremal curve \( E' \) by flat families of curves in surfaces of degree \( s_0(C) \).

**References**


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