THE HILBERT SCHEME $H_{4,-1}$

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1. Introduction

These are notes written for the Torino school. They are meant to supply examples for the theory of locally Cohen-Macaulay space curves developed in the main lectures. I feel the study of $H_{4,-1}$, while simple, is sufficiently rich to illustrate some important general results.

In general, the Hilbert schemes $H_{d,g}$ are very complicated. Thus in order to understand their structure one tries to stratify them using more refined invariants. Usually, one begins considering the numerical invariants defined by the cohomology of the ideal sheaf of $C$: the postulation, the speciality and the Rao function - two of these determine the third, given $d$ and $g$. The speciality is the easiest to deal with, as it depends only on the curve $C$ and the very ample divisor $\mathcal{O}_C(1)$, but not on the linear system defining the embedding in $\mathbb{P}^3$. It has to obey some stringent conditions 3.1. On the other hand, any finitely supported sequence of nonnegative integers arises as the Rao function of a space curve, but only up to translation; and in fact Martin-Deschamps and Perrin [6] discovered $H_{d,g}$ always has an irreducible component made up by curves - \textit{extremal curves} - whose Rao function is the largest possible (for all twists!) for curves of degree $d$ and $g$. Later Nollet [8] proved sharp bounds for the Rao function of a curve of degree $d$ and genus $g$ that is not extremal - curves with this "submaximal" Rao function are called subextremal. Given these limitations, it is easy to determine all possible cohomologies for curves of degree $4$ and genus $-1$.

Unfortunately, fixing all these cohomological invariants is in general not enough to obtain an irreducible family - it is in our case, but one needs to prove it. The strongest general result in this direction is [4, Théorème VII.1.1 p. 133]. It says that the family of curves with a fixed Rao module and a given speciality is irreducible. Note that fixing the Rao module and the speciality is the same as fixing the Rao module and the postulation or the Rao module and the entire cohomology.

Using this machinery, we will be able to show that $H_{4,-1}$ has three irreducible components. We will also see that $H_{4,-1}$ is connected - it is an open problem whether $H_{d,g}$ is always connected. These results are special cases of the main theorems of [10].

2. Description of $H_{4,-1}$

We begin determining all possible spectra (i.e. specialities or $h$-vectors) for curves of degree $4$ and genus $g \geq -1$; this is easy thanks to the restrictions imposed by Theorem 3.1 (connectedness in positive degrees and 1-property).

The spectrum $\{0, 1, 2, 3\}$ is the only one giving $g = 3$. The only "connected" spectrum giving $g = 2$ is $\{0, 1, 2^2\}$ which is ruled out by the 1-property; thus there is no curve of degree $4$ and genus $2$. For $g = 1$ the only possibility for the spectrum is $\{0, 1^2, 2\}$. 

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For $g = 0$ we have two possibilities: $\{0^2, 1, 2\}$ and $\{0, 1^3\}$. These correspond to the generic points of the two irreducible components of $H_{4,0}$, namely, the disjoint union of a line and a plane elliptic curve, and the rational quartic.

For $g = -1$, we again have two possibilities for the spectrum, $\{-1, 0, 1, 2\}$ and $\{0^2, 1^2\}$, but, as we will see, the Hilbert scheme $H_{4,-1}$ has three irreducible components.

Let’s analyze the case $g = -1$. Suppose $C$ is a curve with spectrum $\{-1, 0, 1, 2\}$. Then the Rao function of $C$ “starts” with $n = -1$: $\rho(C)(-1) = 1$. Theorem 4.4 implies that $C$ is an extremal curve (see also “cohomological characterizations of extremal curves” in the section 4). According to [6] the family of extremal curves is an irreducible component of $H_{d,g}$, whenever $H_{d,g}$ is not empty (if one allows ACM extremal curves), and its dimension when $d = 4, g = -1$ is 17. We can sketch a proof of this fact in our particular case. In general, it is not too hard to see that the family of extremal curves of a given degree and genus is irreducible: their Rao function and their spectrum are determined by $d$ and $g$; thus those corresponding to a fixed Rao module form an irreducible family by [4, Théorème VII.1.1 p. 133]. Furthermore, their Rao modules are “extremal Koszul modules”, i.e., the quotient of the polynomial ring $R = \mathbb{P}^4_*(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3})$ by an ideal generated by a regular sequence $\{f_1, f_2, f_3, f_4\}$ where $f_1$, and $f_2$ are linear forms, while the degrees of the forms $f_3$ and $f_4$ are determined by $d$ and $g$. The family of such modules is irreducible, and hence so is the family of extremal curves. In our case, the dimension of the family of extremal curves is 17. We will see that $H_{4,-1}$ contains no other family of dimension greater than 16. Thus the extremal curves must form a component of $H_{4,-1}$.

We can construct the general extremal curve in $H_{4,-1}$ as follows. Pick a smooth conic $P$, and a general line $L$ in the plane of $P$. Then the intersection of $P$ and $L$ consists of two points. It follows that a double line $D$ supported on $L$ cuts on $P$ a scheme of length $m$ with $2 \leq m \leq 4$. Consider the curve $E = P \cup D$. We have $\text{deg}(E) = 4$

$$g(E) = g(P) + g(D) + m - 1 = g(D) + m - 1.$$  

Thus, when $g(D) = -m$, we obtain a curve $E$ in $H_{4,-1}$. Note that $E$ is extremal because it contains $P \cap L$, a planar subcurve of degree $3 = \text{deg}(E) - 1$ (this for example forces the spectrum to contain $\{0, 1, 2\}$).

Let’s compute the dimension of the family of extremal curves constructed this way. Note that the family of double lines $D$ supported on $L$ and having genus $-m$ has dimension $1 + 2m$: this because any such $D$ arises from an exact sequence of the form

$$0 \to \mathcal{I}_D \to \mathcal{I}_L \cong \mathcal{O}_L(-1) \oplus \mathcal{O}_L(-1) \to \mathcal{O}_L(m - 1) \to 0$$

(see [7, 1.4] for details).

When $m = 2$, we count: 8 parameters for the choice of $P$, 2 parameters for the choice of $L$ in the plane of $P$, and $1 + 2m = 5$ for the choice of $D$. Thus the family has dimension 15. When $m = 3$ (resp. $m = 4$), we get 7 (resp. 9) parameters for the choice of $D$, but we have to subtract 1 of each extra intersection we require $D$ to have with $P$. Thus the dimension of the family is 16 for $m = 3$ and 17 for $m = 4$: this means - perhaps surprisingly - that the general extremal curve is given by the above contraction when we require the maximum number of intersections ($m = 4$). We can
also "see" how the case $m = 4$ specializes to the case $m = 3$: we can specialize a double line $D$ of genus $-4$ to a double line $D_0$ of genus $-3$ "plus" an embedded point $A \in P \cap L$; this embedded point accounts for the loss of one of the intersections with $P$.

Now suppose $C$ is curve with spectrum $\{0^2, 1^2\}$. In this case, the spectrum does not determine the Rao function of $C$. The cohomology of $C$ in low degrees looks as follows:

\[
\begin{array}{|c|c|c|c|c|}
\hline
l & h^0\mathcal{I}_C(l) & h^0\mathcal{O}_{\mathbb{P}^3}(l) & h^0\mathcal{O}_C(l) & h^1\mathcal{I}_C(l) \\
\hline
0 & 0 & 1 & 2 & 1 \\
1 & 0 & 4 & 6 & 2 \\
2 & ? & 10 & 10 & ? \\
\hline
\end{array}
\]

We thus have to distinguish two cases depending on whether $C$ is contained in a quadric surface or not - that is, whether $h^1\mathcal{I}_C(2)$ is positive or zero. By Theorem 4.4, if $h^1\mathcal{I}_C(2) > 0$, the curve $C$ is subextremal, and its Rao function is $\rho_{4,-1}^S = \{0, 1^2, 2\}$. By [8] any such curve is obtained from a curve of degree 2 and genus $-2$ by an elementary biliaison of height one on a quadric surface (either the disjoint union of two planes or a double plane). It follows there is a 16 dimensional irreducible family of such curves. The general curve in this family is the disjoint union of 2 conics.

Finally, assume the spectrum of $C$ is $\{0^2, 1^2\}$ and $h^1\mathcal{I}_C(2) = 0$. Then $h^1\mathcal{I}_C(n) = 0$ for $n > 2$ (Munford’s regularity), and the Rao function of $C$ is $\{0, 1^2\}$. In particular, the Rao module $M_C$ has one generator $\alpha$ in degree 0. If $M_C$ has another generator, then $\alpha$ must be killed by three linear forms. But this implies $C$ is contained in a quadric surface - see Corollary 5.5 - contradicting our assumptions on $C$. Hence $M_C$ is monogenous, and up to the choice of coordinates it is isomorphic to $\frac{k[x, y, z, w]}{(x, y, z^2, zw, w^2)}$, which is the Rao module of the disjoint union of a line and a twisted cubic - see [4, Corollaire I.3.5 p. 36]. It follows that any such curve is a specialization with constant cohomology and Rao module of the disjoint union of a line and a twisted cubic [4, Théorème VII.1.1 p. 133]. We conclude these curves form an irreducible family of dimension 16. We thus have:

**Proposition 2.1.** $H_{4,-1}$ has three irreducible components: the 17 dimensional family of extremal curves, and two 16 dimensional components, whose generic curves are respectively the disjoint union of two conics and the disjoint union of a line and a twisted cubic.

We would like to finish proving the connectedness of $H_{4,-1}$. It is true for any $d$ and $g$ for which it makes sense that there are extremal curves in the closure of the family of subextremal curves - see for example [9]. Thus to prove the connectedness of $H_{4,-1}$ it is enough to show how to connect the second and third component. One can do this in several ways (Exercise: prove connectedness using the geometric methods of [1]. Exercise nobody I think has done: prove connectedness using triads). Here I will explain a trick that turned out to be a key in the proof of the connectedness of $H_{4,g}$ for any $g \leq -1$ [10]. It boils down to the following two observations: (1) the two components we want connect both contain a “thick multiplicity four structure” on a
line - in brief, a “thick 4-line”, see below for the definition; (2) the family of thick 4 lines of genus $-1$ is irreducible. (Exercise: the family of extremal curves contains no thick four line, for example because the spectrum of an extremal curve makes it impossible for it to contain a curve of degree 3 and genus 0).

A thick multiplicity four structure on the line $L$ is simply a degree 4 curve supported on $L$ that contains the first infinitesimal neighborhood of $L$ in $\mathbb{P}^3$. If $C$ is such a curve, then $\mathcal{I}_C \subset \mathcal{I}_L^2$ and, since $C$ has degree 4, the quotient $\mathcal{I}_L^2/\mathcal{I}_C$ is a rank one invertible $\mathcal{O}_L$-module. Then $g(C) = -1$ implies this quotient is isomorphic to $\mathcal{O}_L$, so that we have an exact sequence

$$0 \to \mathcal{I}_C \to \mathcal{I}_L^2 \to \mathcal{I}_L^3 \to \mathcal{O}_L \to 0.$$ 

Since

$$\frac{\mathcal{I}_L^2}{\mathcal{I}_L^3} \cong \mathcal{O}_L(-2)^{\oplus 3},$$

we see that giving such a $C$ is the same as giving a surjective morphism $\mathcal{O}_L(-2)^{\oplus 3} \to \mathcal{O}_L$ modulo an automorphism of $\mathcal{O}_L$. Thus the family of thick multiplicity four structures on the line $L$ is parametrized by an open subset of a $\mathbb{P}^8$, and so is irreducible. The exact sequence above implies that the spectrum of $C$ is $\{0^2, 1^2\}$. On the other hand, we may arrange for $\mathcal{I}_C/\mathcal{I}_L^3$ to be isomorphic to either $\mathcal{O}_L(-2) \oplus \mathcal{O}_L(-4)$ or $\mathcal{O}_L(-3) \oplus \mathcal{O}_L(-3)$. In the first case, $C$ is contained in a quadric surface, in the second it is not. Thus we can find thick four lines both in the second and in the third component of $H_{d,-1}$.

3. Properties of the spectrum

The spectrum of a locally Cohen-Macaulay space curve $C$ is by definition the sequence of nonnegative integers

$$h_C(n) = \partial^2 h^0 \mathcal{O}_C(n).$$

Since the Hilbert polynomial of $C$ is linear, one has $h_C(n) = \partial^2 h^1 \mathcal{O}_C(n)$, so the information contained in the spectrum is precisely the speciality of the curve $C$. The degree of $C$ is equal to $\sum h_C(n)$, while the arithmetic genus of $C$ is given by the formula:

$$g(C) = \sum_{n \in \mathbb{Z}} (n - 1) h_C(n) + 1.$$ 

We will write the spectrum of $C$ in the form

$$\{h_1^{h_C(n_1)}, \ldots, h_r^{h_C(n_r)}\}$$

where $n_1, \ldots, n_r$ are the integers for which $h_C(n) > 0$. Finally we suppress the "exponent" $h_C(n)$ if $h_C(n) = 1$. For example, $\{0,1^2\}$ means $h_C(0) = 1$, $h_C(1) = 2$, and $h_C(n) = 0$ if $n \neq 0, 1$.

The index of speciality $e(C)$ of a curve $C$ is by definition the largest integer $n$ such that $h^1 \mathcal{O}_C(n) > 0$. 
Theorem 3.1 ([11], 2.4 and 4.1, [12], 2.3; cf. [14]). Let $C$ be a locally Cohen-Macaulay pure dimensional curve in $\mathbb{P}^3$ with index of speciality $e$. Then $e \geq -2$ and the spectrum of $C$ has the following properties:

- **Connectedness in positive degrees:**
  \[ h_C(n) \geq 1 \text{ for } 0 \leq n \leq e + 2, \quad \text{and} \quad h_C(n) = 0 \text{ for } n > e + 2. \]

- **1-property:**
  Suppose $h_C(l) = 1$ for some integer $l$ with $1 \leq l \leq e + 1$. Then
  \[ h_C(n) = 1 \text{ for } l \leq n \leq e + 2. \]

4. **Extremal and subextremal curves**

Given a curve $C$, the function $\rho_C : \mathbb{Z} \rightarrow \mathbb{Z}$ defined by $\rho_C(n) = h^1(I_C(n))$ is called the Rao function of $C$. It is the Hilbert function of the Rao module $M_C = H^1_*(\mathbb{P}^3, I_C)$ of $C$.

**Theorem 4.1** ([5],[8]). Let $C \subset \mathbb{P}^3$ be a curve of degree $d$ and arithmetic genus $g$. Then

1. $C$ is planar if and only if $g = \frac{1}{2}(d - 1)(d - 2)$. If $C$ is planar, $\rho_C(n) = 0$ for all $n \in \mathbb{Z}$.
2. If $C$ is not planar, then $d \geq 2$ and $g \leq \frac{1}{2}(d - 2)(d - 3)$ and
   \[ \rho_C(n) \leq \rho_{E,d,g}^{E}(n) \text{ for all } n \in \mathbb{Z}. \]

where

\[
\rho_{d,g}^{E}(n) = \begin{cases} 
\max(0, \rho_{d,g}^{E}(n + 1) - 1) & \text{if } n \leq -1, \\
\frac{1}{2}(d - 2)(d - 3) - g & \text{if } 0 \leq n \leq d - 2, \\
\max(0, \rho_{d,g}^{E}(n - 1) - 1) & \text{if } n \geq d - 1.
\end{cases}
\]

**Definition 4.2.** A curve $C \subset \mathbb{P}^3$ is called **extremal** if it has the largest Rao function allowed by its degree and genus. Thus $C$ of degree $d$ and genus $g$ is extremal if $\rho_C = \rho_{E,d,g}^E$ (we set $\rho_{E,d,g} = 0$ when $g = \frac{1}{2}(d - 1)(d - 2)$).

**Remark 4.3.** The definition of an extremal curve in [6], [8] includes the extra condition that the curve be not ACM.

**Geometric description of extremal curves** We can give a geometric description of extremal curves - cf. [6, 8, 2]: a curve is extremal if and only if it is one of the following:

1. $C$ is a plane curve;
2. $C$ has $d = 3$ and $g = 0$;
3. $C$ has $d = 4$ and $g = 1$;
4. $C$ is a non-planar curve which contains a plane curve of degree $d - 1$.

One knows there exists a locally Cohen-Macaulay curve of degree $d$ and genus $g$ in $\mathbb{P}^3$ - i.e., $H_{d,g}$ is nonempty - if and only if either $d \geq 1$ and $g = \frac{1}{2}(d - 1)(d - 2)$ or $d \geq 2$ and $g \leq \frac{1}{2}(d - 2)(d - 3)$. If $d \geq 2$ and $g \leq \frac{1}{2}(d - 2)(d - 3)$, one can construct an extremal curve of degree $d$ and genus $g$ as follows. Pick a plane curve $P$ of degree $d - 2$ and a line $L$ in the plane of $P$. One can choose (exercise) a double structure $D$ on $L$ in such a way that $E = P \cup D$ has genus $g$. Then $E$ has degree $d$, genus $g$ and is extremal because it contains the degree $(d - 1)$ plane curve $P \cup L$. Thus,
whenever nonempty, $H_{d,g}$ contains extremal curves - in fact, they form an irreducible component of $H_{d,g}$ [6].

**Other cohomological characterizations of extremal curves** In terms of the spectrum: a curve of degree $d$ is extremal if and only if its spectrum "contains" $\{0, \ldots, d-2\}$ (which is the spectrum of a plane curve of degree $d-1$). Thus extremal curves are those curves whose spectrum has the form

$$\{0, \ldots, d-2\} \cup \{m\}$$

where either $m = d-1$ or $m \leq 1$.

It follows that $C$ is extremal if and only if $e(C) \geq \deg(C) - 4$. In terms of the postulation, $C$ is extremal if and only if $C$ is contained in pencil of quadric surfaces, i.e., $h^0(\mathbb{P}^3, \mathcal{I}_C(2)) \geq 2$.

**Subextremal curves** Nollet [8], improving results of Ellia, proved sharp bounds for the Rao function of a non-extremal curve (in terms of $d$ and $g$):

**Theorem 4.4** ([8]). Let $C \subset \mathbb{P}^3$ be a curve of degree $d$ and genus $g$ that is not extremal. Then $d \geq 3$ and

1. $g \leq \frac{1}{2}(d-3)(d-4)+1$; if equality holds, then $d \geq 5$ and $C$ is ACM;
2. if $d \geq 4$, then

$$\rho_C(n) \leq \rho_{d,g}^S(n) \quad \text{for all } n \in \mathbb{Z}$$

where

$$\rho_{d,g}^S(n) = \begin{cases} 
\max(0, \rho_{d,g}^S(n+1) - 1) & \text{if } n \leq 0, \\
\frac{1}{2}(d-3)(d-4)+1-g & \text{if } 1 \leq n \leq d-3, \\
\max(0, \rho_{d,g}^S(n-1) - 1) & \text{if } n \geq d-2.
\end{cases}$$

**Definition 4.5.** A curve $C \subset \mathbb{P}^3$ of degree $d \geq 4$ and genus $g$ is called subextremal if it is not extremal and $\rho_C = \rho_{d,g}^S$. Again, this differs from the terminology of [8] in that we include among subextremal curves those ACM curves of degree $d \geq 5$ which have genus $1/2(d-3)(d-4)+1$.

**Remark 4.6.** There exist subextremal curves if $d = 4$ and $g \leq 0$ and if $d \geq 5$ and $g \leq \frac{1}{2}(d-3)(d-4)+1$.

**Proposition 4.7** ([8, 2]). Let $C \subset \mathbb{P}^3$ be a curve of degree $d \geq 4$ and genus $g$. The following are equivalent:

1. $C$ is subextremal;
2. $C$ is obtained from an extremal non-planar curve by an elementary biliaison of height one on a quadric surface;
3. either $C$ is ACM and $(d, g) \in \{(5, 2), (6, 4)\}$, or $C$ is a divisor of type $(1, 3)$ on a smooth quadric surface, or there is a plane $H$ such that $\mathcal{I}_{C \cap H, H} = \mathcal{I}_{Z, H}(2-d)$ with $Z$ zero-dimensional contained in a line, and the residual scheme $\text{Res}_H(C)$ to the intersection of $C$ with $H$ is a plane curve of degree two.

5. A Technical Result from [13]

Let $R = k[x, y, z, w]$ be the homogeneous coordinate ring of $\mathbb{P}^3$. All the $R$ modules we consider in this section are graded and finitely generated. For any two such modules $M$ and $N$, the module $\text{Hom}_R(M, N)$ is graded, its homogeneous elements of
degree \( l \) being the homogeneous homomorphisms of degree \( l \). It follows that the modules \( \text{Ext}^l_R(M, N) \) are graded. All the morphism we will consider will be homogeneous, usually of degree zero, unless otherwise specified.

**Lemma 5.1.** Let \( D \) be an arithmetically Cohen-Macaulay curve in \( \mathbb{P}^3 \) and let \( \Omega_D = \text{Ext}_R^2(R_D, R(-4)) \) be the canonical module of \( D \). Let \( J \) be a homogeneous ideal in \( R \). There is an isomorphism of graded \( R \)-modules:

\[
\text{Ext}^1_R(\Omega_D, R/J) \cong \frac{J \cap I_D(4)}{J \cdot I_D(4)}
\]

**Proof** Consider a free graded resolution of \( R_D \) over \( R \):

\[
0 \to F_2 \xrightarrow{\beta} F_1 \xrightarrow{\alpha} R \to R_D \to 0
\]

Dualizing and twisting by \( R(-4) \) we obtain a resolution of \( \Omega_D \):

\[
0 \to R(-4) \xrightarrow{\alpha^\vee} F_1^\vee(-4) \xrightarrow{\beta^\vee} F_2^\vee(-4) \to \Omega_D \to 0
\]

We compute \( \text{Ext}^1_R(\Omega_D, R/J) \) applying the functor \( \text{Hom}_R(\Omega, -) \) to the sequence 2. For a free graded module \( F \) we have \( \text{Hom}_R(F^\vee, R/J) \cong R/J \otimes F \), thus we find

\[
\text{Ext}^1_R(\Omega_D, R/J) \cong \frac{\text{Ker}}{\text{Im}} \left( 1 \otimes \alpha(4) : R/J \otimes F_1(4) \to R/J \otimes R(4) \right)
\]

This together with free resolution 1 gives

\[
\text{Ext}^1_R(\Omega_D, R/J) \cong \text{Tor}_1^R(R/J, R_D)(4) \cong \frac{J \cap I_D(4)}{J \cdot I_D(4)}
\]

(the last isomorphism being a standard exercise in homological algebra).

**Proposition 5.2.** Let \( C \) and \( D \) be two curves in \( \mathbb{P}^3 \) and assume that \( D \) is arithmetically Cohen-Macaulay. Suppose there is a nonzero homogeneous homomorphism of degree \( l \)

\[
\phi : \Omega_D \to M_C(l)
\]

from the canonical module of \( D \) to the Rao module of \( C \). Then either \( C \) and \( D \) have a common component, or there is a surface of degree \( l + 4 \) which contains both \( C \) and \( D \) and whose equation is not in \( I_C \cdot I_D \).

**Proof** We apply the functor \( \text{Hom}_R(\Omega_D, -) \) to the exact sequence

\[
0 \to R_C \to A_C \to M_C \to 0
\]

to obtain the exact sequence:

\[
\text{Hom}_R(\Omega_D, A_C) \to \text{Hom}_R(\Omega_D, M_C) \to \text{Ext}^1_R(\Omega_D, R_C).
\]

By assumption the group in the middle is nonzero in degree \( l \), so at least one of the other two groups is nonzero in degree \( l \). If \( \text{Hom}_R(\Omega_D, A_C) \) is nonzero, then \( C \) and \( D \) have a common component because:

\[
\text{Supp}(\text{Hom}_R(\Omega_D, A_C)) = \text{Ass}(A_C) \cap \text{Supp}(\Omega_D) = \text{Ass}(C) \cap \text{Ass}(D).
\]

On the other hand, assume that \( \text{Ext}^1_R(\Omega_D, R_C) \) is nonzero in degree \( l \). Then by Lemma 5.1

\[
\frac{I_C \cap I_D}{I_C \cdot I_D}
\]
is nonzero in degree \(l + 4\).

**Remark** We can use the same argument anytime we have an exact sequence

\[ 0 \rightarrow R_C \rightarrow B \rightarrow N \rightarrow 0, \]

where \(B\) is an \(R\) module with \(\text{Ass}(B) = \text{Ass}(R_C)\), together with a nonzero homogeneous homomorphism \(\Omega_D \rightarrow N\); the case we have in mind is \(B = \Omega_C(n)\).

**Corollary 5.3.** Let \(C\) be a curve in \(\mathbb{P}^3\). Suppose there is a nonzero \(m \in H^1 I_C(t)\) which is killed by two independent linear forms \(z\) and \(w\). Then either \(C\) contains the line \(L\) of equations \(z = w = 0\), or \(C\) lies on a surface of degree \(t + 2\).

**Proof** Mapping \(1\) to \(m\) we get a nonzero morphism of degree \(t\) from \(R_L\) to \(M_C\). Since \(\Omega_L \cong R_L(-2)\), the statement follows from Proposition 5.2.

**Corollary 5.4.** Let \(C\) be a curve in \(\mathbb{P}^3\). Suppose that for some integer \(t \geq 0\) we have \(h_C(n) = \begin{cases} n + 1 & \text{if } -1 \leq n \leq t - 1 \\ t + 2 & \text{if } n = t, t + 1 \end{cases}\) Then either \(C\) contains a line in its support or \(s(C) \leq t + 2\).

**Proof** Assume \(s(C) \geq t + 2\). Then our assumption on the spectrum implies that \(h^1 I_C(t) = 1\) and \(h^1 I_C(t + 1) = 2\); thus there is a nonzero element \(m \in H^1 I_C(t)\) which is killed by two linearly independent linear forms. Now the statement follows from Corollary 5.3.

**Remark** Using the Remark following Proposition 5.2 with \(B = \Omega_C(-e)\), one can prove variants of the previous Corollary. For example, one can assume \(h_C(e + 2 - n) = \begin{cases} n + 1 & \text{if } 0 \leq n \leq t - 1 \\ t + 2 & \text{if } n = t, t + 1 \end{cases}\) and still conclude that either \(C\) contains a line in its support, or \(s(C) \leq t + 2\).

**Example** Suppose \(C\) is an integral curve of degree 8 satisfying \(\omega_C \cong O_C(1)\). Then \(e(C) = 1\) and by duality \(h_C(l) = h_C(3 - l)\) for all integers \(l\). Hence the spectrum of \(C\) is \(\{0, 1^3, 2^3, 3\}\) thus \(C\) satisfies the hypothesis of the Corollary with \(t = 1\). As \(C\) is integral, we conclude that \(C\) is contained in a surface of degree three. Note that \(h^0 O_C(3) = 20\), thus there is no such curve of maximal rank in \(\mathbb{P}^3\). This was already proven by Hartshorne and Sols in [3], Lemma 1.2, by a similar method.

We can construct a curve \(C\) with the same spectrum, but not lying on any cubic surface as follows: let \(D\) be the disjoint union of a line and a twisted cubic curve and let \(C\) be the curve obtained from \(D\) by an elementary double link of height one on a quartic surface.

**Corollary 5.5.** Let \(C\) be a curve in \(\mathbb{P}^3\). Suppose there is a nonzero \(m \in H^1 I_C(t)\) which is killed by three linearly independent linear forms. Then \(s(C) \leq t + 2\).

**Proof** For each line \(L\) whose equations kill \(m\) we get a morphism of degree \(t\) from \(R_L\) to \(M_C\), so either \(C\) contains \(L\) or \(s(C) \leq t + 2\). By assumption, there is a one dimensional family of such lines, and \(C\) cannot contain all of them. Hence \(s(C) \leq t + 2\).
REFERENCES