LOWER-ORDER PERTURBATIONS OF CRITICAL GROWTH NONLINEARITIES IN SEMILINEAR ELLIPTIC EQUATIONS*

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Abstract. The solvability of semilinear elliptic equations with nonlinearities in the critical growth range depends on the terms with lower-order growth. We generalize some known results to a wide class of lower-order terms and prove a multiplicity result in the left neighborhood of every eigenvalue of $-\Delta$ when the subcritical term is linear. The proofs are based on variational methods; to assure that the considered minimax levels lie in a suitable range, special classes of approximating functions having disjoint support with the Sobolev "concentrating" functions are constructed.

1. Introduction. In this paper we consider the semilinear problem

$$\begin{align*}
-\Delta u &= g(x, u) + |u|^{2^*-2}u \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega,
\end{align*}
$$

where $\Omega \subset \mathbb{R}^n$ ($n \geq 3$) is an open bounded domain with smooth boundary $\partial \Omega$, $2^* = \frac{2n}{n-2}$ is the critical Sobolev exponent, and $g(\cdot, s)$ has subcritical growth at infinity (i.e., $\lim_{|s| \to \infty} \frac{g(s)}{|s|^{2^*}} = 0$). Consider the Hilbert space $H := H^1_0(\Omega)$ endowed with the Dirichlet scalar product. Let $\lambda_k$, $k \in \mathbb{N}$, be the eigenvalues of $-\Delta$ relative to the homogeneous Dirichlet problem in $\Omega$; it is well-known that each eigenvalue has finite multiplicity $\mu_k$, and that $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots$ with $\lambda_k \to +\infty$ as $k \to \infty$. We denote by $\sigma(-\Delta) = \{\lambda_k\}$ the spectrum of $-\Delta$.

Define the functional $J : H \to \mathbb{R}$ by

$$J(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx - \int_{\Omega} G(x, u) \, dx - \frac{1}{2^*} \int_{\Omega} |u|^{2^*} \, dx,$$

where $G(x, s) = \int_0^s g(x, t) \, dt$; if $g$ is continuous we have $J \in C^1(H, \mathbb{R})$ and the critical points of the functional $J$ correspond to (weak) solutions of equation (1).
However, the standard variational arguments do not apply because the imbedding $H \subset L^2(\Omega)$ is not compact; i.e., the functional $J$ does not satisfy the Palais-Smale condition (115) (PS condition). And indeed, for equations with critical growth, nontrivial solutions may not exist: if $\Omega$ is strictly star-shaped and $\lambda \leq 0$, then by a result of Pohožaev ([16]) the following equation has only the trivial solution $u \equiv 0$:

$$
\begin{cases}
-\Delta u = \lambda u + u|u|^{2^* - 2} & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
$$

(2)

In recent years the situation of "lack of compactness" has been studied extensively, and has led to a good understanding of the underlying phenomena. Let $S$ denote the best constant of the imbedding $H \subset L^2(\Omega)$ (see [20]), in a pioneering result Brezis-Nirenberg ([8]) showed that the functional $J$ corresponding to (2), i.e., $J(u) = \frac{1}{2} \int |\nabla u|^2 - \frac{\lambda}{2} \int u^2 - \frac{1}{2} \int |u|^{2^*}$, satisfies the PS condition in the interval $(0, \frac{S^2}{\lambda})$, but fails to satisfy it at the level $\frac{S^2}{\lambda}$ (see [12] for an alternative proof). The reason for the failure of the PS condition at level $\frac{S^2}{\lambda}$ is explained by a profound result of Struwe ([18]): consider the "limiting problem"

$$
\begin{cases}
-\Delta u = u|u|^{2^* - 2} & \text{in } \mathbb{R}^n \\
u(x) \to 0 & \text{as } |x| \to \infty
\end{cases}
$$

(3)

the representation result of [18] states, roughly speaking, that if $(u_m) \subset H$ is a PS sequence for $J$, then it can be approximated (for $m \to \infty$) by a solution $u_1$ of (2) plus a certain number $k$ of (suitably rescaled) solutions $u_j^\varepsilon$ ($j = 1, \ldots, k$) of (3). The crucial fact is that positive solutions of (3), which realize the constant $S$ in the imbedding $H^1(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$, are unique up to rescaling, and they yield an increase of the level of the functional by $\frac{S^2}{\lambda}$. In view of this, a possible strategy for proving existence results for equations of form (1) consists in constructing minimax levels for the functional $J$ which lie in the interval $(0, \frac{S^2}{\lambda})$; indeed, this is the method proposed in [8], and which we will use here. However, we note that with our assumptions on the lower-order term $g$ the functional $J$ may have negative critical levels, and that the above interval may not be a "range of compactness" for $J$. But we will prove that a PS sequence within the above energy range yields a nontrivial solution of (1) by means of its weak limit.

We recall the following known existence results for equations (1) and (2):

i) The first celebrated existence result for equation (2) is due to Brezis-Nirenberg ([8]) who proved that if $\lambda \in (\lambda_1, \lambda_2)$ (with $\lambda^* = 0$ if $n = 4$, and $0 < \lambda^*(\Omega) < \lambda_1$ if $n = 3$), then there exists a nontrivial positive solution of equation (2) (equations of more general form are also considered in [8]; see Section 2).

ii) Later, Cerami-Fortunato-Struwe ([12]) established the existence of a bifurcation from any eigenvalue $\lambda_k$ of $-\Delta$ and estimated the left neighborhood $(\lambda_k - \varepsilon, \lambda_k)$ of $\lambda_k$ in which the bifurcation branch can be extended; more precisely, they showed that for $\nu = S(\Omega)^{-2/n}$, where $S$ is the best Sobolev imbedding constant, the number of (pairs of) solutions of equation (2) is at least $m(\lambda) = \sum_{k \in K} \mu_k$ with $K := \{k \in \mathbb{N} : \lambda < \lambda_k < \lambda + \nu\}$.

iii) In a subsequent paper, Capozzi-Fontanato-Palmieri ([10]) proved that equation (2) admits nontrivial solutions for all $\lambda > 0$ if $n \geq 5$, and for all $\lambda > 0$ with $\lambda \notin \sigma(-\Delta)$ if $n = 4$ (see the remark after Corollary 1 below); see also [3] for an alternative proof in the case where $\lambda \notin \sigma(-\Delta)$. For the case $n = 3$ the existence of a nontrivial solution for all $\lambda > \frac{\lambda_3}{2}$ has been obtained by Comte ([14]) in the particular case that $\Omega$ is a ball.

The aims of this paper are

a) to generalize the results i) and ii) to a broader class of subcritical perturbations of the critical growth term $|u|^{2^* - 2}u$ in equation (2); in particular, we extend the results of [8] to the case where the functional $J$ has a generalized mountain pass structure ([17]) (i.e., $\lambda_1 \leq \lim_{|u| \to \infty} \frac{\int_{\Omega} \frac{2(|u|^2_2 - |\nabla u|^2_2)}{|u|^{2^* - 2}}}{\int_{\Omega} |u|^2}$ instead of the classical mountain pass geometry ([2]) (i.e., $0 \leq \lim_{|u| \to \infty} \frac{\int_{\Omega} \frac{2(|u|^2_2 - |\nabla u|^2_2)}{|u|^{2^* - 2}}}{\int_{\Omega} |u|^2} \leq \lambda_1$), and the result of [10] to the case where a more general function $g(x, u)$ replaces the term $\lambda u$ in equation (2).

b) to show that the "bifurcation branch" starting in $\lambda_{k+1}$ can be continued up to $\lambda_k - \delta_k$ for some $\delta_k > 0$. This yields a multiplicity result: for $\lambda \in (\lambda_k - \delta_k, \lambda_k)$ there exist at least two (pairs of) solutions of equation (2); in particular, for $\lambda \in (\lambda_1 - \delta_1, \lambda_1)$, we find a "small" positive solution (the Brezis-Nirenberg solution) and a "large" solution which changes sign.

c) to extend the multiplicity result of Cerami-Fortunato-Struwe ([12]) for odd nonlinearities when $n \geq 5$: if the multiplicity of $\lambda_k$ is $\mu_k$, then there exists $\delta_k > 0$ such that if $\lambda \in (\lambda_k - \delta_k, \lambda_k)$ then (2) has at least $2(\mu_k + 1)$ nontrivial solutions.

The basic idea of our approach is the following: since we are looking for solutions of equation (2) with $\lambda \geq \lambda_1$, the functional $J$ has a generalized mountain-pass structure; this requires (in order to prove that the minimax level stays below $\frac{S \pi^2}{n}$) an estimate of the maximum of the functional $J$ over subsets of $V \oplus \mathbb{R}^n \{u\}$, where $V$ is some finite-dimensional subspace of $H$ and $u$ is the concentrating (under rescaling) function mentioned above. Since $V$ and $u$ are not orthogonal, this estimate involves "mixed terms" which are difficult to estimate (see [10]). However, an "orthogonalization" (in $H$ as well as in $L^p$) of these functions can be obtained by replacing the space $V$ by a space $V'_\varepsilon$ consisting of functions which approximate the functions in $V$, but are zero where $u_\varepsilon$ is nonzero, that is, by disjointing their supports. Of course, now the approximation error of $V'_\varepsilon$ must be estimated, but this can be handled more easily than the mentioned mixed terms.

2. Statement of the results. We make the following assumption on the subcritical term $g$:

$$
g : \Omega \times \mathbb{R} \to \mathbb{R} \text{ is a Carathéodory function with}
$$

$$
\forall s > 0 \exists \varepsilon_s \in L^{\frac{2^*}{2}}(\Omega) \text{ such that } |g(x, s)| \leq \alpha_s(x) + \varepsilon_s|s|^{\frac{2^*}{2}}
$$

(4)

for a.e. $x \in \Omega$ and $\forall s \in \mathbb{R}$.
The other assumptions are imposed on the primitive $G(x, s) = \int_0^s g(x, t) \, dt$: we first assume that
\begin{equation}
G(x, s) \geq 0 \quad \text{for a.e. } x \in \Omega, \quad \forall s \in \mathbb{R}.
\end{equation}
(5)

Note that the lower-order perturbation $g(x, u)$ is allowed to change sign.

For the further assumptions we distinguish two cases.

2.1. Nonresonance near the origin. Assume that there exist $k \in \mathbb{N}$, $\delta > 0$, $\sigma > 0$, and $\mu \in (\lambda_k, \lambda_{k+1})$ such that
\begin{equation}
\frac{1}{2} (\lambda_k + \sigma) s^2 \leq G(x, s) \leq \frac{1}{2} \mu s^2 \quad \text{for a.e. } x \in \Omega, \quad \forall |s| \leq \delta;
\end{equation}
(6)
furthermore, we assume that
\begin{equation}
G(x, s) \geq \frac{1}{2} (\lambda_k + \sigma) s^2 - \frac{1}{2} \sigma |s|^{2^*} \quad \text{for a.e. } x \in \Omega, \quad \forall s \neq 0.
\end{equation}
(7)

As a consequence of (4), (6) we have $g(x, 0) = 0$ for almost every $x \in \Omega$ and therefore $u = 0$ solves (1).

In $\mathbb{R}^3$ we also need a growth condition at infinity:
\begin{equation}
\lim_{s \to +\infty} \frac{G(x, s)}{s^4} = +\infty \quad \text{uniformly with respect to } x \in \Omega_0.
\end{equation}
(8)

The reason for the additional assumption (8) in the case $n = 3$ will become clear in the proof of Lemma 5; a similar condition is considered in [8].

With the above assumptions we will prove

**Theorem 1.** For $n \geq 4$ assume (4)-(7), for $n = 3$ assume (4)-(8); then equation (1) admits a nontrivial solution.

**Remarks.** 1. When $n \geq 4$ this theorem generalizes the result of [10] (for $\lambda \notin \sigma(-\Delta)$) and of [3], where the particular case $g(x, s) = \lambda s$ is considered.

2. The case $\mu \in (0, \lambda_1)$ has been studied in [8], Corollaries 2.1 and 2.2. See also [11] for a different class of perturbations. In the forthcoming paper [4] the existence of solutions for $\mu \in (0, \lambda_1)$ is shown imposing only conditions (4), (5) and (6) (with $\lambda_0 = -\infty$). As already mentioned, these assumptions allow $g(x, s)$ to change sign. We refer also to [1] for the existence of positive solutions in the ball with $g$ which changes sign.

3. Theorem 1 remains true replacing condition (5) by a weaker assumption which allows $G$ also to change sign; see the Remark in Section 4.

4. When $n = 3$ our result extends Corollary 2.3 in [8] as we do not assume that $0 < \mu < \lambda_1$ nor $g(x, s) \geq 0$; of course we do not obtain a positive solution.

2.2. Resonance near the origin. Let us now consider the situation where $\frac{\partial G(x, 0)}{\partial s}$ is "possibly an eigenvalue" in a neighborhood of $s = 0$; i.e., assume that there exist $\delta > 0$ and $\mu \in (\lambda_k, \lambda_{k+1})$ such that
\begin{equation}
\frac{1}{2} \mu \delta^2 \leq G(x, s) \leq \frac{1}{2} \mu s^2 \quad \text{for a.e. } x \in \Omega, \quad \forall |s| \leq \delta.
\end{equation}
(9)

In this case we need to move $\sigma$ from the quadratic part to the critical part in condition (6) and impose a growth condition like (8) for all dimensions. More precisely, we assume that there exists $\sigma \in (0, 1/2^*)$ such that
\begin{equation}
G(x, s) \geq \frac{1}{2} \lambda_k s^2 - \left( \frac{1}{2^*} - \sigma \right) |s|^{2^*} \quad \text{for a.e. } x \in \Omega, \quad \forall s \in \mathbb{R};
\end{equation}
(10)
morover, we require that
\begin{equation}
\lim_{s \to +\infty} \frac{G(x, s)}{s^{2^*/(2^*-1)}} = +\infty \quad \text{uniformly with respect to } x \in \Omega_0.
\end{equation}
(11)

We will prove the following

**Theorem 2.** Let $n \geq 3$ and assume (4), (5), (9)-(11); then equation (1) admits a nontrivial solution.

Note that for $G(x, s) = \frac{1}{2} \lambda s^2$ condition (11) is satisfied for $n \geq 5$ (more precisely, for $n^2 - 4n - 4 > 0$ which may be of interest in view of radial solutions ([5])); therefore, from Theorems 1 and 2 we obtain

**Corollary 1.**

1. If $n \geq 5$ then equation (2) admits a nontrivial solution $\forall \lambda > 0$.
2. If $n = 4$ then equation (2) admits a nontrivial solution $\forall \lambda > 0$ such that $\lambda \notin \sigma(-\Delta)$.

**Remark.** This result is proved in [10]; it was pointed out to us by H. Brezis that the proof of Theorem 0.1 in [10] does not work if $n = 4$ and $\lambda \in \sigma(-\Delta)$: indeed the crucial estimate (2.10a) does not hold for $\bar{u}^*$.

2.3. Multiplicity results in high dimensions. We also obtain a multiplicity result for equation (2): we prove that if $n \geq 5$ then the bifurcation branch starting from the eigenvalue $\lambda_{k+1} \in \sigma(-\Delta)$, $k \geq 1$, can be extended up to a left neighborhood of $\lambda_k$.

**Theorem 3.** Let $n \geq 5$; then, $\forall \lambda_k \in \sigma(-\Delta)$, there exists $\delta_k > 0$ such that $\lambda \in (\lambda_k - \delta_k, \lambda_k)$ equation (2) admits at least $\mu_k + 1$ pairs of nontrivial solutions.

This result yields in particular a multiplicity result in the interval $(\lambda_1 - \delta_1, \lambda_1)$:
Corollary 2. Let \( n \geq 5 \); then there exists a \( \delta_1 \in (0, \lambda_1) \) such that in \( (\lambda_1 - \delta_1, \lambda_1) \) there exist at least two solutions, the "small" positive solution of Brezis-Nirenberg, and a "large" solution which changes sign.

The above corollary is already known if \( n \geq 6 \), for \( \delta_1 = \lambda_1 \), see [13, 21], but the result seems new in the case \( n = 5 \); furthermore, we point out that the solutions found here are below the energy level \( \frac{1}{2} S^{n/2} \), while in [13] this may not be the case: there a "higher" energy estimate is obtained, and so our solutions (in the case \( n \geq 6 \)) may be different from the ones found in [13].

Remark. Theorem 3 and Corollary 2 can be easily extended to equations of the more general form (1).

3. The variational characterization. The proofs of our results involve variational techniques as in [7, 8, 10, 12, 19]. As we deal with infinitesimal quantities we make use of the following Landau symbols: we denote

\[
\begin{align*}
    f(x) &= o(g(x)) & \text{as } x \to x_0 & \text{if } \lim_{x \to x_0} \frac{f(x)}{g(x)} = 0 \\
    f(x) &= O(g(x)) & \text{as } x \to x_0 & \text{if } \limsup_{x \to x_0} \left| \frac{f(x)}{g(x)} \right| < +\infty.
\end{align*}
\]

In this section we describe the variational characterization, which is based on a linking argument. In the standard variational procedure the PS condition is crucial; it can be stated as follows: A sequence \( \{u_m\} \subset H \) is called a PS sequence for \( J \) at level \( c \) if \( J(u_m) \to c \) and \( J'(u_m) \to 0 \) in \( H^{-1} \). The functional \( J \) satisfies the PS condition at level \( c \), if every PS sequence at level \( c \) has a convergent subsequence in \( H \).

As already mentioned, the functional \( J \) does not satisfy the PS condition; however, the following result holds:

Lemma 1. Assume (4) and let \( \{u_m\} \subset H \) be a PS sequence for \( J \); then there exists \( u \in H \) such that \( u_m \rightharpoonup u \) up to a subsequence and \( J(u) = 0 \). Moreover, if \( J(u_m) \rightharpoonup c \) with \( c \in (0, \frac{S^{n/2}}{n}) \) then \( u \neq 0 \) and hence \( u \) is a nontrivial solution of (1).

Proof. The proof is standard (see [8]); we sketch it. Let

\[
\begin{align*}
    f(x, s) &= g(x, s) + |s|^{p-2}s \\
    F(x, s) &= \int_0^s f(x, t) \, dt;
\end{align*}
\]

and since (4) holds, we have

\[
\exists \theta \in (0, \frac{1}{2}) \exists \varepsilon > 0 \text{ such that } F(x, s) \leq \theta f(x, s)s \text{ for a.e. } x \in \Omega \quad \forall |s| \geq \varepsilon.
\]

therefore \( \{u_m\} \) is bounded (see [2]) and \( \exists u \) such that \( u_m \rightharpoonup u \) up to a subsequence. Furthermore, \( J'(u) = 0 \) by weak continuity of \( J \); see [18].

Assume \( c \in (0, \frac{S^{n/2}}{n}) \) and, for the sake of contradiction, \( u = 0 \); as the term \( g(x, u_m)u_m \) is subcritical, by Theorem 2.27 in [9] we infer from \( J'(u_m)[u_m] = o(1) \) that

\[
\|u_m\|^2 - \|u_m\|_{2^*}^2 = o(1).
\]

By the definition of \( S \), \( \|u\|^2 \geq S \|u\|_2^2 \), for all \( u \in H \), we obtain

\[
\alpha(1) \geq \|u_m\|^2(1 - S^{-2/3} \|u_m\|_{2^*}^{-2}).
\]

If now \( \|u_m\| \to 0 \) we contradict \( \alpha > 0 \); therefore, \( \|u_m\|^2 \geq S^{n/2} + o(1) \) and by (12) we get

\[
J(u_m) = \frac{1}{n} \|u_m\|^2 + \frac{n - 2}{2n} (\|u_m\|_2^2 - \|u_m\|_{2^*}^2) + o(1)
\]

\[
\geq \frac{1}{n} S^{n/2} + o(1),
\]

which contradicts \( c < \frac{1}{n} S^{n/2} \).

Remark. If in Lemma 1 we also assume that

\[
\frac{1}{2} g(x, s)s - G(x, s) + \frac{1}{n} |s|^{p-2} s \geq 0 \quad \forall s \in \mathbb{R} \quad \text{for a.e. } x \in \Omega
\]

then every PS sequence \( \{u_m\} \) is relatively compact. Indeed, if \( u \) is a critical point for \( J \), then \( J'(u)[u] = 0 \), which substituted into \( J(u) \) gives

\[
J(u) = \frac{1}{2} \int_\Omega g(x, u)u - \int_\Omega G(x, u) + \frac{1}{n} \int_\Omega |u|^{p-2} u \geq 0;
\]

that is, all the critical levels of \( J \) are positive; then, by reasoning as in [13], we obtain that either \( \{u_m\} \) is relatively compact or \( c \geq \frac{1}{n} S^{n/2} \).

By Lemma 1, to prove Theorems 1–3 it suffices to build a PS sequence for \( J \) at a level strictly between 0 and \( \frac{S^{n/2}}{n} \).

Denote by \( e \) an \( L^2 \) normalized eigenvector relative to \( \lambda_1 \in \sigma(-\Delta) \), let \( H^+ \) denote the space spanned by the eigenvectors corresponding to the eigenvalues \( \lambda_1, \ldots, \lambda_k \) and \( H^- := (H^+)^\perp \), and let \( P_k : H \to H^- \) denote the orthogonal projection.

Without restrictions assume that \( 0 \in \Omega \), take \( m \) large enough so that \( B_{2/m} \subset \Omega \) where \( B_{2/m} \) denotes the ball of radius \( 2/m \) with center in \( 0 \); in the case of assumption (8) (respectively (11)) assume also that \( 0 \in \Omega_0 \) and take \( m \) so large that \( B_{2/m} \subset \Omega_0 \).

Consider the functions \( \zeta_m : \Omega \to \mathbb{R} \) defined by

\[
\zeta_m(x) := \begin{cases} 
0 & \text{if } x \in B_{1/m} \\
\frac{1}{m} |\cdot| - 1 & \text{if } x \in A_m = B_{2/m} \setminus B_{1/m} \\
1 & \text{if } x \in \Omega \setminus B_{2/m}.
\end{cases}
\]
Then, define the following “approximating eigenfunctions” \(e_i^n := \xi_{mn} e_i\) and the space
\[ H_m^n := \operatorname{span}[e_i^n; i = 1, \ldots, k]. \]
We first prove that the functions \(e_i^n\) converge to the eigenfunctions \(e_i\) and we estimate the approximation error:

**Lemma 2.** As \(m \to \infty\) we have
\[ e_i^n \to e_i \quad \text{in } H \quad \text{and} \quad \max_{\{u \in H_m^n; \|u\| = 1\}} \|u\|^2 \leq \lambda_k + c_k m^{2-\alpha}. \]

**Proof.** We have
\[
\int_\Omega |\nabla (e_i^n - e_i)|^2 = \int_\Omega |\nabla \xi_{mn} + (\xi_{mn} - 1)\nabla e_i|^2 \\
= \int_{A_m} |\nabla \xi_{mn}|^2 e_i^2 + 2 \int_{A_m} (\nabla \xi_{mn})(\xi_{mn} - 1)e_i \nabla e_i + \int_{B_m} (\xi_{mn} - 1)^2 |\nabla e_i|^2 \\
\leq c \|e_i\|^2 m^{2-\alpha} + c \|\nabla e_i\|_\infty \|e_i\|_{L^1} m^{1-\alpha} + c \|\nabla e_i\|_2^2 m^{2-\alpha},
\]
where \(c > 0\) is some positive constant independent of \(e_i\).

Next, denote \(\partial B := \{u \in H : \int_\Omega u^2 = 1\}\), and let \(u \in H \cap \partial B\); i.e., \(u = \sum_{j=1}^J a_j e_j\) with \(\sum_{j=1}^J a_j^2 = 1\). Note that then \(v_m := \xi_{mn} u = \sum a_j e_j^m = \sum a_j e_j^n\);
\(\text{i.e., } v_m \in H_m^n\). For each \(v \in H \cap \partial B\) let \(a_m(v) := \|v_m\|_{L^1}^2\) (so that \(a_m(v) = 1\) whenever \(v \in \partial B\)). We first show
\[ 1 \geq a_m(v) \geq 1 + c \|v\|_\infty^2 m^{-\alpha}; \]
this follows from
\[
1 = \int_\Omega a_m(v) v^2_m \\
= a_m(v) (1 - \int_{B_m} v^2 + \int_{A_m} \xi_{mn} v^2) \\
\geq a_m(v) (1 - c \|v\|_\infty^2 m^{-\alpha}).
\]
Let now \(\bar{u}_m \in H_m^n \cap \partial B\) such that \(\|\bar{u}_m\|^2 = \max_{u \in H_m^n \cap \partial B} \|u\|^2\). Then
\[ \bar{u}_m = a_m(\bar{u}) \sum_j a_j e_j^m = a_m(\bar{u}) \xi_{mn} \sum_j a_j e_j = a_m(\bar{u}) \xi_{mn} \bar{u}, \]
and hence, arguing as in (13),
\[
\|\bar{u}_m\|^2 = a_m(\bar{u}) \int_\Omega |\nabla (\xi_{mn} \bar{u})|^2 \\
\leq (1 + c \|\bar{u}\|_{L^1}^2 m^{-\alpha}) \int_\Omega |\nabla \xi_{mn} + \xi_{mn} \nabla \bar{u}|^2 \\
\leq (1 + c \|\bar{u}\|_{L^1}^2 m^{-\alpha}) (c \|\bar{u}\|_{L^1}^2 m^{2-\alpha} + c \|\nabla \xi_{mn}\|_{L^1} \|\bar{u}\|_{L^1} m^{1-\alpha} + \|\bar{u}\|^2) \\
\leq c \|\bar{u}\|_{L^1}^2 m^{2-\alpha} + c \|\nabla \bar{u}\|_{L^1} \|\bar{u}\|_{L^1} m^{1-\alpha} + \lambda_k \\
\leq c \|\bar{u}\|_{L^1}^2 m^{2-\alpha} + \lambda_k.
\]

As in [8], we consider the family of functions
\[
u_{\epsilon}^{\alpha}(x) := \frac{[\epsilon(n-2)\xi_{mn}]^{\alpha/2}}{[\epsilon^2 + |x|^2]^{\alpha/2}}, \quad (\epsilon > 0),
\]
which solve (3) and satisfy \(\|\nu_{\epsilon}^{\alpha}\|^2 = \|u_{\epsilon}^{\alpha}\|^2 = S^{n/2}\) for all \(\epsilon > 0\). Define a positive cut-off function \(\eta \in C_0^\infty(B_{1/2})\) such that \(\eta = 1\) in \(B_{1/2}\), \(\eta \leq 1\) in \(B_{1/4}\) and \(\|
abla \eta\|_{L^\infty} \leq 4\delta\); consider the sequence of functions \(u_{\epsilon}^{\alpha}(x) := \eta(\alpha x - u_{\epsilon}^{\alpha}(x))\). As \(\epsilon \to 0\) we have the following estimates which are due to Brezis-Nirenberg ([8]) (see also Lemma 2.1 in [10] and page 164 in [19]):
\[
\|u_{\epsilon}\|^2 = S^{n/2} + O(\epsilon^{n-\alpha}) \quad \|u_{\epsilon}\|_{L^2}^2 = S^{n/2} + O(\epsilon^{\alpha}).
\]

For \(u \in H_m^n \cap \partial B\) we now write \(u = w + a u_{\epsilon}\), where by definition
\[ \text{supp}(u_{\epsilon}) \cap \text{supp}(w) = \emptyset. \]

Several remarks are now in order:

\((R_1)\) If (4) and either (6) or (9) hold then there exist \(\alpha, \rho > 0\) such that
\[ J(v) \geq \alpha \quad \forall v \in \partial B_{\rho} \cap \partial H^+. \]

Indeed, by the assumptions we infer that there exists \(C > 0\) such that
\[ G(x, s) \leq \frac{1}{2} \mu s^2 + C|s|^2 \]
and therefore \(J(v) \geq c_1 \|v\|^2 - c_2 |v|^2\) for all \(v \in \partial H^+\).

\((R_2)\) Define
\[ Q_{\mu} := \{(B_{R} \cap H_m^n) \cap [0, R][u_{\epsilon}]\}; \]
then \(\exists R > \rho\) such that \(\max_{v \in Q_{\mu}} J(v) \leq \omega_m\) with \(\omega_m \to 0\) as \(m \to \infty\).

Indeed, by (7) we clearly have
\[
\lim_{m \to \infty} \max_{u \in H_m^n} J(u) = 0
\]
and by (5)
\[ J(ru_{\epsilon}) \leq \frac{1}{2} r^2 \|u_{\epsilon}\|^2 - \frac{1}{2r^2} r^2 \|u_{\epsilon}\|_{L^2}^2, \]
which, by (15), becomes negative if \(r = R\) and \(R\) is large enough; therefore, \(J(v) \leq \omega_m\) for all \(v \in (H_m^n \cap (B_{R} \cap H_m^n))\); finally, since \(\omega_{\rho<\frac{1}{2}} J(ru_{\epsilon}) \to < +\infty\), if \(v \in \{B_{R} \cap H_m^n \cap [0, R][u_{\epsilon}]\}\), by (16) we obtain \(J(v) \leq 0\) for large enough \(R\).

\((R_3)\) The functional \(J\) satisfies all the assumptions of the linking theorem [17] except for the PS condition.
Indeed, in view of Lemma 2 we have: if \( m \) is large enough, then

\[
P_k H^*_m = H^- \quad \text{and} \quad H^- \otimes H^+ = H,
\]

where \( P_k : H \rightarrow H^- \) is the projection introduced above; therefore, \( \partial B_p \cap H^+ \) and \( \partial I^+_p \) link (cf. [17]).

The next sections are devoted to the construction of PS sequences at level \( c < \frac{S^{n/2}}{n} \), which, by Lemma 1, yield a nontrivial solution of (1).

We will proceed as follows: Let

\[
\Gamma := \{ h \in C(\mathcal{Q}_m, H) : h(\nu) = \nu, \forall \nu \in \partial I^+_m \};
\]

then by standard methods we obtain a PS sequence for \( J \) at level

\[
c = \inf_{h \in \Gamma} \max_{\nu \in \mathcal{Q}_m} J(h(\nu)).
\]

Moreover, since the identity \( I_d \in \Gamma \), we have

\[
c \leq \max_{\nu \in \mathcal{Q}_m} J(\nu);
\]

Theorems 1–3 follow if we can prove that for \( \varepsilon \) small enough

\[
sup_{\nu \in \mathcal{Q}_m} J(\nu) < \frac{1}{n} S^{n/2}.
\]

4. Proof of Theorem 1. In this section we prove Theorem 1; so, assume hypotheses (4)–(7) if \( n \geq 4 \), (4)–(8) if \( n = 3 \) and choose \( m \) large enough so that

\[
c_k m^{2-n} < \sigma
\]

where \( c_k \) is as in Lemma 2 and \( \sigma \) is as in (6).

We claim that (17) holds; for the sake of contradiction assume that

\[
\forall \varepsilon > 0 \quad \sup_{\nu \in \mathcal{Q}_m} J(\nu) \geq \frac{1}{n} S^{n/2}.
\]

As the set \( \{ \nu \in \mathcal{Q}_m : J(\nu) \geq 0 \} \) is compact, the supremum in (19) is attained. Therefore, for all \( \varepsilon > 0 \) there exist \( w_\varepsilon \in H^- \) and \( t_\varepsilon \geq 0 \) such that, for \( u_\varepsilon := w_\varepsilon + t_\varepsilon u_\varepsilon \), we have

\[
J(u_\varepsilon) = \max_{\nu \in \mathcal{Q}_m} J(\nu) \geq \frac{1}{n} S^{n/2};
\]

that is,

\[
\frac{1}{2} \Vert u_\varepsilon \Vert^2 - \int_\Omega G(x, u_\varepsilon) - \frac{1}{2n} \Vert v_\varepsilon \Vert_2^2 \geq \frac{1}{n} S^{n/2}, \quad \forall \varepsilon > 0.
\]

Let us establish the main properties of the sequences \( \{ t_\varepsilon \} \) and \( \{ w_\varepsilon \} \); by (R2) in the previous section we immediately obtain that the sequences \( \{ t_\varepsilon \} \subset \mathbb{R}^+ \) and \( \{ w_\varepsilon \} \subset H^- \) are bounded. Hence, up to subsequences we may assume that

\[
t_\varepsilon \rightarrow t_0 \geq 0 \quad \text{and} \quad w_\varepsilon \rightarrow w_0 \in H^-.
\]

where the convergence of \( \{ w_\varepsilon \} \) can be viewed in any norm since the space \( H^- \) is finite dimensional. As \( w_\varepsilon \in H^- \), by using Lemma 2, (7) and (18) we have

\[
J(w_\varepsilon) \leq \frac{\lambda_k + c_k m^{2-n}}{2} \Vert w_\varepsilon \Vert_2^2 - \int_\Omega G(x, w_\varepsilon) - \frac{1}{2n} \Vert w_\varepsilon \Vert_2^2 \leq \frac{c_k m^{2-n} - \sigma}{2} \Vert w_\varepsilon \Vert_2^2 \leq 0.
\]

Furthermore, we have

Lemma 3. If (20) holds; then \( t_0 = 1 \).

Proof. As \( G \) has subcritical growth at infinity and as \( \{ t_\varepsilon \} \) is bounded, we get

\[
\lim_{\varepsilon \rightarrow 0} \int_\Omega G(x, t_\varepsilon u_\varepsilon) = 0;
\]

hence, by (15) we obtain

\[
J(t_\varepsilon u_\varepsilon) \leq S^{n/2} \left( \frac{\tau^2}{2} - \frac{\tau^2}{2n} \right) + o(1),
\]

which combined with (16) and (21) gives

\[
J(u_\varepsilon) = J(w_\varepsilon) + J(t_\varepsilon u_\varepsilon) \leq \phi(t_0) \cdot S^{n/2} + o(1),
\]

where \( \phi(x) = \frac{x^2}{2} - \frac{x^2}{2n} \). To conclude note that

\[
\max_{x \geq 0} \phi(x) = \phi(1) = \frac{1}{n} \quad \text{and} \quad \phi(x) < \frac{1}{n} \quad \forall x \geq 0, \ x \neq 1;
\]

thus, if \( t_0 \neq 1 \) then we contradict (20) for \( \varepsilon \) small enough. \( \square \)

Next, we estimate the main order terms in \( J(t_\varepsilon u_\varepsilon) \):
Lemma 4. As \( \varepsilon \to 0 \) we have
\[
\frac{1}{2} \| t_{\varepsilon} u_{\varepsilon} \|^{2} - \frac{1}{2n} \| t_{\varepsilon} u_{\varepsilon} \|_{2}^{2n} \geq - \frac{1}{n} S^{\varepsilon/2} + O(\varepsilon^{n-2}).
\]

Proof. Using (15) we get (as \( \varepsilon \to 0 \))
\[
\frac{1}{2} \| t_{\varepsilon} u_{\varepsilon} \|^{2} - \frac{1}{2} \| t_{\varepsilon} u_{\varepsilon} \|_{2}^{2n} \geq \frac{1}{2} S^{\varepsilon/2} + \frac{1}{2n} \| t_{\varepsilon} u_{\varepsilon} \|_{2}^{2n} + O(\varepsilon^{n-2})
\]

hence,
\[
\frac{1}{2} \| t_{\varepsilon} u_{\varepsilon} \|^{2} - \frac{1}{2} \| t_{\varepsilon} u_{\varepsilon} \|_{2}^{2n} \geq \frac{1}{2} S^{\varepsilon/2} + \frac{n-2}{2n} (t_{\varepsilon} u_{\varepsilon}^{2} - 1)S^{\varepsilon/2} + O(\varepsilon^{n-2}).
\]

To conclude, note that \( \max_{x \in \mathbb{R}} |x^{2} - 1 - \frac{n-2}{n} (x^{2} - 1)| = 0. \)

Finally, we estimate the lower-order term \( \int_{\Omega} G(x, t_{\varepsilon} u_{\varepsilon}) \):  

Lemma 5. There exists a function \( \tau = \tau(\varepsilon) \) such that \( \lim_{\varepsilon \to 0} \tau(\varepsilon) = +\infty \) and such that for \( \varepsilon \) small enough we have
\[
\int_{\Omega} G(x, t_{\varepsilon} u_{\varepsilon}) \geq \tau(\varepsilon) \cdot \varepsilon^{n/2}.
\]

Proof. Let us first consider the case \( n = 3 \): for \( \varepsilon \) small enough we have \( B_{\varepsilon} \subset B_{1/2m} \subset \Omega_{0} \) and by (14)
\[
\int_{\Omega} G(x, t_{\varepsilon} u_{\varepsilon}) \geq \int_{B_{\varepsilon}} G(x, t_{\varepsilon} [3\varepsilon^{2}]^{1/4} x^{2} + |x|^{2}]^{1/2}).
\]

By (8) we infer that \( \exists \bar{s} > 0 \) and an increasing function \( \varphi = \varphi(s) \) with
\[
\lim_{s \to +\infty} \varphi(s) = +\infty
\]

such that if \( s \geq \bar{s} \) then
\[
G(x, s) \geq \varphi(s) s^{4} \quad \text{for a.e. } x \in \Omega.
\]

Next, note that if \( \varepsilon \) is small enough we have (recall that \( t_{\varepsilon} \to 1 \) by Lemma 3)
\[
t_{\varepsilon} [3\varepsilon^{2}]^{1/4} \geq \bar{s}, \quad \forall x \in B_{\varepsilon};
\]

hence, by (22),
\[
\int_{\Omega} G(x, t_{\varepsilon} u_{\varepsilon}) \geq c \int_{B_{\varepsilon}} \varphi_{1/2} \cdot \varepsilon^{n/2} \geq c \varphi_{1/2} \cdot \varepsilon \cdot \varepsilon^{n/2}.
\]

and the assertion follows by taking \( \tau(\varepsilon) = c \varphi_{1/2} \cdot \varepsilon^{n/2} \).

Let us now prove the result in the case \( n \geq 4 \). A straightforward calculation yields
\[
t_{\varepsilon} u_{\varepsilon}^{2} = \gamma \quad \Longleftrightarrow \quad |x| = \Phi(\gamma) = (t_{\varepsilon} \cdot \gamma)^{2/n-2} \sqrt{n(n-2) \cdot \varepsilon - \gamma^{2}}^{1/2};
\]

then, there exists \( c_{1} > 0 \) such that, for \( \varepsilon \) small enough, we have \( \Phi(\delta) < c_{1} \sqrt{\varepsilon} < (2n)^{-1} \) where \( \delta \) is as in (6); hence, by (5) and (6) we have
\[
\int_{\Omega} G(x, t_{\varepsilon} u_{\varepsilon}) \geq c \int_{\Phi(\delta)} u_{\varepsilon}^{4} \cdot \varepsilon^{n-1} dr \geq c \int_{c_{1} \sqrt{\varepsilon}}^{1/nm} \varepsilon^{n-1} \cdot \delta^{n-1} dr
\]

and therefore
\[
\int_{\Omega} G(x, t_{\varepsilon} u_{\varepsilon}) \geq c \varepsilon^{n-1} \int_{c_{1} \sqrt{\varepsilon}}^{1/nm} r^{n-1} dr
\]

Thus, the result is proved for all \( n \geq 3 \). \( \square \)

The proof of Theorem 1 is now easily completed: by (16), (21) and Lemmas 4 and 5 (which hold because we assumed (20)) we have
\[
J(u_{0}) = J(u_{0}) + J(t_{\varepsilon} u_{\varepsilon}) \leq \frac{1}{n} S^{n/2} + (c - \tau(\varepsilon)) \cdot \varepsilon^{n/2},
\]

which contradicts (20) for \( \varepsilon \) small enough; thus (17) holds.

Remark. If \( n \geq 4 \) assumption (5) can be weakened as follows:
\[
\exists \gamma > 0, \exists \bar{s} < \frac{n}{n-2} \text{ such that } G(x, s) \geq -\gamma |s|^{n}, \text{ for a.e. } x \in \Omega, \forall s \in \mathbb{R}. \quad (5')
\]

Then one has

Theorem 1'. Assume (4), (5'), (6), (7). If \( n = 4 \), then equation (1) has a nontrivial solution. If \( n \geq 5 \), (1) has a nontrivial solution provided that \( \gamma = \gamma(\alpha) \) in (5') is sufficiently small.

Proof. The only change in the proof concerns the estimates (23) and (24): by (5') and (6) we obtain
\[
\int_{\Omega} G(x, t_{\varepsilon} u_{\varepsilon}) \geq c \int_{c_{1} \sqrt{\varepsilon}}^{1/2m} \varepsilon^{n-2} \cdot \varepsilon^{n-1} dr - c \gamma \int_{0}^{c_{1} \sqrt{\varepsilon}} \left( \frac{\varepsilon^{n-2}}{(\varepsilon^{2} + r^{2})^{n-2}} \right)^{n} r^{n-1} dr,
\]
and therefore
\[
\int_{\Omega} G(x, t_x u_x) \geq c e^{n/2} \int_{t_x \sqrt{\epsilon}}^{c \sqrt{\epsilon}} r^{3-n} dr - c y e^{n/2} \int_{c \sqrt{\epsilon}}^{c \sqrt{\epsilon}} r^{-n-1} dr,
\]
and finally
\[
\int_{\Omega} G(x, t_x u_x) \geq \begin{cases} 
  c e^{n/2} \cdot e^{2-n/2} - c y e^{n/2} & \text{if } n \geq 5 \\
  c e^{2} \cdot |\log \epsilon| - c y e^{2} & \text{if } n = 4.
\end{cases}
\]

With these estimates we again obtain (25), which completes the proof.

5. Proof of Theorem 2. The proof of Theorem 2 follows the same lines as that of Theorem 1; however, some refinements of the estimates are required. In this section we assume hypotheses (4), (5), (9)–(11). In order to emphasize the dependence on \( m \) we denote \( v^m, u^m, w^m \) instead of \( u_x, w_x, v_x \) (this dependence is “hidden” in the cut-off function \( \eta \)).

As in the previous section we want to show (17) and we reason by contradiction assuming that (19) holds: for all \( m \) large enough (say \( m \geq \tilde{m} \)) and all \( \epsilon > 0 \) there exist \( v^\epsilon_m \in Q^m \) and \( t_x \geq 0 \) such that
\[
\frac{1}{2} \|v^\epsilon_m\|^2 - \int_{\Omega} G(x, v^\epsilon_m) - \frac{1}{2} \|v^\epsilon_m\|^2_{L^2} \geq \frac{1}{n} S^{n/2} \quad \forall \epsilon > 0, \quad \forall m \geq \tilde{m}.
\]

If (26) holds, then the sequences \( \{t_x\} \) and \( \{w^\epsilon_m\} \) satisfy again
\[
t_x \geq c > 0 \quad \text{and} \quad \|w^\epsilon_m\| \leq c.
\]

The following lemma contains a refinement of the estimates (15) which highlights the dependence on \( m \):

Lemma 6. Let \( m \to \infty \) and assume that \( \epsilon = \epsilon(m) = o(1/m) \); then
\[
\|u^\epsilon_m\|^2 = S^{n/2} + O((\epsilon m)^{n/2-2}), \quad \|u^\epsilon_m\|^2_{L^2} = S^{n/2} + O((\epsilon m)^n).
\]

Moreover, if (11) holds, then there exists a function \( \phi \) such that \( \lim_{x \to +\infty} \phi(x) = +\infty \) and
\[
\int_{\Omega} G(x, u^\epsilon_m) \geq e^\epsilon(n-2)/(n+2) \phi(e^{-1}).
\]

Proof. As \( \|\nabla \eta\| \leq 4m \), the proof of the estimates \( \|u^\epsilon_m\|^2 = S^{n/2} + O((\epsilon m)^{n/2-2}) \) and \( \|u^\epsilon_m\|^2_{L^2} = S^{n/2} + O((\epsilon m)^n) \) can be obtained as for (15); see [8].

By (11) there exists an increasing function \( \tau \) such that \( \lim_{x \to +\infty} \tau(x) = +\infty \) satisfying \( G(x, s) \geq \tau(s) \cdot s^8/n^4 \) for almost every \( x \in \Omega_0 \) and for all \( s \geq \tilde{s} \) (for a suitable \( \tilde{s} > 0 \)); therefore, reasoning as in the proof of Lemma 5 (in the case \( n = 3 \)) we obtain
\[
\int_{\Omega} G(x, t_x u_x^m) \geq c \int_0^\tau \left( \frac{e^{-\epsilon(n-2)/2}}{(s^2 + r^2)^{n-2}} \right)^{8n/(n-2)} \cdot \tau(s)^{2n/(n-2)} \cdot r^{n-1} dr
\]
\[
geq c e^{-\epsilon(n-2)/(n+2)} \cdot \tau(\epsilon m)^n \cdot \tau^8 \cdot \tau^2 \cdot r^{n-1} dr
\]
\[
geq c e^{-\epsilon(n-2)/(n+2)} \cdot \tau(\epsilon m)^n \cdot \tau^2 \cdot (n+2)
\]

and the result follows by setting \( \phi(x) = c\sqrt{x} e^{-x/(n+2)} \).

We now choose a suitable \( \epsilon = \epsilon(m) \) in order to deal only with the parameter \( m \), and to arrive at a contradiction to (26) for \( m \) large enough: we take
\[
e(\epsilon(m)) = m^{-n/(n+2)};
\]

therefore, as \( m \to \infty \), \( \epsilon(m) = o(1/m) \) and Lemma 6 applies. From now on, we denote by \( v^m, u^m, w^m \) the functions \( v^\epsilon_m, u^\epsilon_m, w^\epsilon_m \) with the above choice of \( \epsilon \) and with \( t_x \) the corresponding \( t_x \).

We first estimate \( J(t_x u^m) \): by Lemma 6 we infer

Lemma 7. If \( m \) is large enough we have
\[
J(t_x u^m) \leq \frac{1}{n} S^{n/2} - cm^{n(n-2)/2} \phi(cm^{n(n+2)/2}),
\]

where \( \phi \) is the function defined in Lemma 6.

Proof. For the moment assume just that \( \epsilon = \epsilon(m) = o(1/m) \) as \( m \to \infty \); then by Lemma 6, (27) and by reasoning as in the proof of Lemma 4 we obtain
\[
J(t_x u^m) \leq \frac{1}{n} S^{n/2} + O((\epsilon m)^{n-2}) - e^{\epsilon(n-2)/(n+2)} \phi(e^{-1}).
\]

With the choice of \( \epsilon \) as in (28) we get
\[
J(t_x u^m) \leq \frac{1}{n} S^{n/2} + O((\epsilon m)^{n-2}) - cm^{n(n-2)/2} \phi(cm^{n(n+2)/2}),
\]

and hence
\[
J(t_x u^m) = J(t_x u^m) \leq \frac{1}{n} S^{n/2} - cm^{n(n-2)/2} \phi(cm^{n(n+2)/2}).
\]

Finally, we estimate the part of the functional relative to \( u^m \):
Lemma 8. If \( m \) is large enough we have

\[
J(w^m) \leq cm^{(2-n)/2}.
\]

Proof. By (10) and Lemma 2 we get (for large \( m \))

\[
J(w^m) \leq \frac{1}{2} \|w^m\|^2 - \frac{\lambda_\ast}{2} \|w^m\|_2^2 - \sigma \|w^m\|_2^2 \leq c_1 (m^{2-n})^2 - c_2 \|w^m\|_2^2.
\]

Consider the function \( h(x) = c_1 x^{2-n} \cdot x^2 - c_2 x^{2n/(n-2)} \); its derivative vanishes for \( x = cm^{(a-2)/4} \) and therefore, using (27), we have \( h(\|w^m\|_2) \leq cm^{(2-n)/2} \), with this and (29) we obtain the result.

The proof of Theorem 2 is now obtained by (16) and Lemmas 7 and 8:

\[
J(w^m) = J((m^*) w^m) + J(w^m)
\]

\[
\leq \frac{1}{n} S^{n/2} - \frac{1}{2} \|w^m\|_2^2 - \frac{1}{2} \|w^m\|_2^2 - \frac{1}{2} \|w^m\|_2^2.
\]

for \( m \) sufficiently large. This contradicts (26), and the proof of Theorem 2 is complete.

6. Proof of Theorem 3. For all \( \lambda > 0 \) define

\[
J_{\lambda}(u) := \frac{1}{2} \|u\|^2 - \frac{1}{2} \|u\|_2^2 - \frac{1}{2} \|u\|_2^2.
\]

Let \( k \in \mathbb{N} \) and choose \( \varepsilon(m) \) as in (28): by Lemmas 7 and 8 we infer that there exists a function \( \delta(m) \) with \( \lim_{m \to \infty} \delta(m) = 0 \) and \( m \in \mathbb{N} \) such that

\[
\delta(m) > 0 \quad \forall m \geq \tilde{m} \quad \text{and} \quad \sup_{u \in \mathcal{O}_{\lambda}} J_{\lambda}(u) < \frac{1}{n} S^{n/2} - \delta(m);
\]

here we write \( Q_m \) instead of \( Q^m \) since \( \varepsilon = \varepsilon(m) \) is given. Therefore, there exists \( \tilde{m} \geq \tilde{m} \) such that

\[
\sup_{m \geq \tilde{m}} \delta(m) = \delta(\tilde{m}) =: \tilde{\delta} > 0
\]

and

\[
\sup_{u \in \mathcal{O}_{\lambda}} J_{\lambda}(u) \leq \frac{1}{n} S^{n/2} - \tilde{\delta}.
\]

(30)

As the set \( \mathcal{Q}_{\lambda} \) is compact, we have

\[
\sup_{u \in \mathcal{Q}_{\lambda}} \frac{1}{2} \|u\|_2^2 = \gamma < \infty.
\]

(31)

Let \( \delta := \tilde{\delta}/\gamma \) and let \( \lambda \in (\lambda_k - \delta, \lambda_k) \); consider the spaces

\[
V = \text{span}(e_i^{\lambda_k} : i = 1, \ldots, k) \oplus \mathbb{R}[u_k] \quad W = \text{span}(e_i : i \geq k).
\]

As \( \lambda < \lambda_k \) we obviously have

\[
\exists \alpha, \beta > 0 \text{ such that } J(u) > \alpha, \quad \forall u \in \delta B_p \cap W;
\]

moreover, by (30) and (31) we have

\[
\sup_{u \in \mathcal{Q}_{\lambda}} J_{\lambda}(u) \leq \frac{1}{n} S^{n/2} - \tilde{\delta} + (\lambda_k - \lambda)\gamma < \frac{1}{n} S^{n/2}.
\]

To complete the proof of Theorem 3 it now suffices to apply Theorem 2.4 in [6] (see also the restatement in [12]) with \( b = \frac{1}{\alpha} S^{n/2} \) and using that \( \dim V - \text{codim} W = \mu_k + 1 \).

REFERENCES