Second International Symposium on “Riordan Arrays and Related Topics”

Lecco, Politecnico di Milano, Italy, July 14-16, 2015

In honor of Lou Shapiro and Renzo Sprugnoli

Book of Abstracts

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Lectures

40 minutes lectures

- ◇ Emeric Deutsch, Jumps and jump-length in ordered trees
- ◇ Lou W. Shapiro, The Riordan Group – 2015
- ◇ Renzo Sprugnoli, Characterization and history of Riordan arrays

30 minutes lectures

- ◇ Aoife Hennessy, Bijections of weighted lattice paths using Riordan array decompositions
- ◇ Hana Kim, Polynomials and Riordan matrices
- ◇ István Mező, Evaluation of some finite and infinite sums by the r-Whitney numbers

Jumps and jump-length in ordered trees

Emeric Deutsch

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In the preorder traversal of an ordered tree, any transition from a node at a deeper level to a node on a strictly higher level is called a *jump*; the positive difference of the levels is called *jump distance*; the sum of the jump distances in a given tree is called the *jump-length*. The statistics (i) number of jumps and (ii) jump-length will be investigated on ordered trees and on various subclasses of ordered trees.

The Riordan Group – 2015

Lou W. Shapiro

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The outline for this talk is as follows.

First we will survey of some recent main results and new applications. Among these new applications are connections with shift register sequences, Somos-4 sequences with elliptic curves, Fermat's theorem, Banach fixed points, the Double Riordan group, Lie Alebras, and the g-A equivalence.

Next we discuss pictures, ordered trees, subgroup structure, the $V = TL$ equation, and semidirect products.

We conclude by looking at future directions and some key open questions.

Characterization and history of Riordan arrays

Renzo Sprugnoli

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My talk will be divided into two parts, one concerning the diagonals top-left to bottom-right of Riordan arrays, and the other a short history of these matrices. It is important to realize that the results obtained can be extended to recursive matrices, which seems the most natural setting where these properties should be studied.

Riordan arrays have been studied from many points of view, corresponding to different characterizations and construction methods. Their classical definition is $D = \mathcal{R}(d(t),\ h(t))$, where $d(t)$ (the *boundary value*) is the generating function of column 0, and $h(t)$ (the *recurrence rule*) allows to pass from any column to the successive. This definition privileges an approach based on columns, while if one prefers a row based study, the A-sequence reveals to be more appropriate. Unfortunately, at least to my knowledge, an approach through diagonals has never been tried.

Some properties of Riordan arrays can be easily observed by this approach. For example, diagonal 0 is a geometric progression, often having ratio $q = 1$ as in the Pascal triangle. Analogously, the diagonal 1 contains an arithmetic-geometric progression, but as we pass to the other diagonals, things become more complex and their structure is difficult to analyze.

In this part of my talk, I'll discuss some results obtained on diagonals of Riordan arrays. Several other points remain at a conjecture level, and I thank in advance every one, who will take a part in this research with hints or suggestions of every kind.

The vast literature on Riordan arrays, piled up in few years, evidences that these matrices cover important sectors of Mathematics (Combinatorial Analysis, but also parts of Algebra and Geometry), are easily handled and furnish effective results. In particular, the subset of all Riordan arrays having $d(0) \neq 0$, $h(0) = 0$ and $h'(0) \neq 0$, constitutes a group, the *Riordan Group* of *proper* Riordan arrays, having as operation the usual row-by-column product, in this context, however, corresponding to simple manipulations of algebraic formulae.

These concepts were formulated at the beginning of the last decade of the second millennium, but the authors were well aware that analogous ideas had been discussed starting 130 years earlier. In fact, the French Mathematician Blissard introduced what he called *Umbral Calculus*, a formalism that shadows (the Latin *Umbra* is *Shadow* in English) some algebraically complex constructs with simpler ones. This calculus was used extensively by Lucas, Bell and eventually by G.-C. Rota and S. Roman, who gave a modern and rigorous foundation to the formalism.

In the formulation of Rota-Roman, the Riordan group is just the 1-Umbral Calculus, but this latter approach did not receive so much attention by researchers. Why? I'll dedicate the last part of my talk to explain the reasons of this strange fact, and some of them can be listed in this Abstract: the notation different from the one currently used in Combinatorics; the apparent limitation of applications restricted to orthogonal polynomials; the emphasis given to the exponential case; and others, less relevant.

Bijections of weighted lattice paths using Riordan array decompositions

Aoife Hennessy

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Waterford Institute of Technology

This talk concerns paths counted by Riordan arrays arising from the decomposition of certain Hankel matrices and bijective relationships between them. We consider certain Hankel matrices, H under two decompositions, $H = L_M D L_M^T$ with L_M a Riordan array of generating functions that count weighted Motzkin paths and a $H = L_L BDB L_L^T$ decomposition, with L_L a Riordan array with generating functions that count weighted Łukasiewicz paths. A bijection is introduced between these paths.

References

[1] P. Flajolet, *Combinatorial aspects of continued fractions*, Discrete Mathematics **32** (1980), 125–161.
[2] A. Hennessy, *A study of Riordan srrays with applications to continued fractions, orthogonal polynomials and lattice paths*, PhD Thesis, Waterford Institute of Technology, 2011.
[3] F. Lehner, *Cumulants, Lattice paths and orthogonal polynomials*, Discrete Mathematics **270** (2003), 177–191.

Polynomials and Riordan matrices

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(Joint work with Gi-Sang Cheon)

Many polynomial sequences have (generalized) Riordan matrices as their coefficient matrices. The structure of a Riordan matrix and the Riordan group is a great help in studying those polynomials. In the first part of this talk, we briefly review such cases which lead to the generating function, recurrence relations, determinantal formula, combinatorial interpretations of the polynomials, and a relationship to other polynomials.

The second part is devoted to our recent work [1] on the zeros of polynomials. A key idea is representing a polynomial as the characteristic polynomial of certain matrix similar to the companion matrix of the polynomial.

References

[1] G.-S. Cheon, H. Kim, *Representing polynomials as characteristic polynomials via the Stieltjes transform*, Linear Algebra Appl. **476** (2015), 184–196.

Evaluation of some finite and infinite sums by the r-Whitney numbers

István Mező

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Nanjing, P. R. China

(Joint work with José L. Ramírez)

In the talk we introduce the r-Whitney numbers through their combinatorial interpretations, and then we show how to use them to evaluate some finite and infinite sums. The main tool is a master theorem deduced through Riordan arrays.

At the end of the talk we mention an open problem with respect to a generalization of the Eulerian numbers.

Contributed talks

- ◇ José Agapito Ruiz, A Catalan array and gamma-numbers
- ◇ Paul Barry, On a transformation of Riordan arrays
- ◇ Naiomi Cameron, On the number of hills among generalized Dyck paths
- ◇ Xi Chen, Total Positivity of Riordan Arrays
- ◇ Gi-Sang Cheon, The Riemann Zeta function and associated polynomials
- ◇ Tian-Xiao He, Row Sums and Alternating Sums of Riordan arrays
- ◇ I-Chiau Huang, Multivariate Riordan bases
- ◇ Sung-Tae Jin, Multidimensional Riordan arrays
- ◇ Ji-Hwan Jung, On the multiple orthogonal polynomials of Sheffer type
- ◇ Sooyeong Kim, The Riordan group in several variables
- ◇ Ana Luzón, Construction of involutions in the Riordan group
- ◇ Manuel Alonso Morón, q-cones: A toy example on Combinatorics and Topology of simplicial complexes
- ◇ Massimo Nocentini, Patterns in Riordan arrays
- ◇ Luis Felipe Prieto-Martínez, The derived series of the Riordan group
- ◇ Minho Song, Krylov matrices versus orthogonal polynomials
- ◇ Fatma Yesil, q-Analogue of Riordan Representation

A Catalan array and gamma-numbers

José Agapito Ruiz

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Gamma-numbers are the coefficients arising in the expansion of a polynomial in terms of a particular basis. If a polynomial is palindromic (symmetric), some of these numbers are necessarily zero, whereas the others may be positive, negative, or zero. Gamma-numbers are especially interesting when they are positive, since positivity implies that the polynomial is palindromic and unimodal. In addition, gamma-numbers may count interesting combinatorial objects. The Eulerian polynomials and the Narayana polynomials are examples of well-known polynomials that have positive gamma-numbers. In this talk I will present a general formula to compute the gamma-numbers of any palindromic polynomial. This new formula involves a Catalan array that is familiar in the theory of Riordan arrays. Conditions for gamma-positivity will also be discussed.

On a transformation of Riordan arrays

Paul Barry

Waterford Institute of Technology

Let $a_{n,k}$ denote the general element of a Riordan array A. For every non-negative integer r we define a new array $A^{(r)}$ with general term

$$a_{n,k}^{(r)} = \sum_{i=0}^{n+r} \binom{n+r}{i} a_{n,i+k}.$$

We show that the matrix $A^{(r)}$ is a Riordan array, and we characterize its A-sequence in terms of that of A. We illustrate these results in the case of some well known Riordan arrays.

On the number of hills among generalized Dyck paths

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(Joint work with Jillian McLeod, US Coast Guard Academy, New London, CT, USA)

It is well known that Dyck paths without hills (that is, ground level subpaths of the form UD) are counted by the Fine numbers, with generating function [2]

$$F(z) = \frac{1-\sqrt{1-4z}}{3z - z\sqrt{1-4z}}\,.$$

The main object of study in this paper is a generalization of the Fine number sequence. We let $F_t(z)$ denote the generating function for the number of generalized Dyck paths without hills (that is, minimal length subpaths of the form $U^{t-1}D$). When $t=2$, $F_t(z)$ corresponds to the Fine number sequence and when $t=3$, $F_t(z)$ gives the number of ternary paths with no hills; that is, paths from $(0,0)$ to $(3n,0)$, using steps of the form $U=(1,1)$ and $D=(1,-2)$, never going below the x-axis and having no hills.

We consider a question posed by L. Shapiro circa 2009: "What is the asymptotic proportion of Dyck paths having an even number of hills?" We provide an answer to this question for generalized Dyck paths, finding in the case when $t=3$ that the probability of a ternary path having an even number of hills approaches 125/169 as path length approaches infinity. We give a limiting distribution for the number of hills among generalized Dyck paths. We also consider a generalization of the following identity:

$$(2z+z^2)F^2(z) - (1+2z)F(z) + 1 = 0\,,$$

for which a combinatorial proof was provided in [1]. We generalize the result to $F_t(z)$ in the following way

$$1 + zF_t^t(z) = \sum_{k=0}^{t-1}\left[\binom{t-1}{k} + \binom{t}{k+1}z\right](-1)^k z^k F_t^{k+1}(z)$$

and explore various combinatorial interpretations of the results.

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Total Positivity of Riordan Arrays

Xi Chen

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(Joint work with Huyile Liang and Yi Wang)

An infinite matrix is called totally positive if its minors of all orders are nonnegative. A nonnegative sequence $(a_n)_{n\geq 0}$ is called log-convex (log-concave, resp.) if $a_i a_{j+1} \geq a_{i+1} a_j$ ($a_i a_{j+1} \leq a_{i+1} a_j$, resp.) for $0 \leq i < j$.

The object of this talk is to study various positivity properties of Riordan arrays, including the total positivity of such a matrix, the log-convexity of the 0th column and the log-concavity of each row. We present sufficient conditions for the total positivity of Riordan arrays. As applications, two classes of special Riordan arrays are considered. One is related to the recursive matrices introduced by M. Aigner [1], the other is the consistent Riordan arrays introduced in [2]. We show that many well-known combinatorial triangles are totally positive and many famous combinatorial numbers are log-convex in a unified approach. In addition, we give a combinatorial interpretation of the log-convexity of the 0th column of a Riordan array.

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The Riemann Zeta function and associated polynomials

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(Joint work with Hana Kim)

The Riemann zeta function

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$$

is defined for complex numbers s with $\mathrm{Re}(s)>1$. The Riemann hypothesis asserts that all nontrivial zeros of $\zeta(s)$ lie on the critical line $\sigma=\frac{1}{2}$. Riemann also extended $\zeta(s)$ to all complex numbers $s\neq 1$ by deriving the analytic continuation using the theta series

$$\theta(x)=\sum_{n=-\infty}^{\infty}\mathrm{e}^{-\pi n^2 x}.$$

In this talk, we introduce a family of polynomials $p_n^{(m,\alpha,\beta)}(s)$ for real numbers m,α,β that satisfies the analog of the Riemann hypothesis. Specifically, we show that all the zeros of $p_n^{(m,\alpha,\beta)}(s)$ lie on the line $\mathrm{Re}(s)=\frac{m}{2}$. Further, using Riordan array technique we show that the polynomials $p_n^{(m,\alpha,\beta)}$ lead to a generalization of the Riemann's integral

$$\int_0^{\infty}\psi(x)x^{\frac{s}{2}-1}\,\mathrm{d}x$$

arising in the Riemann's second proof of the analytic continuation of $\zeta(s)$ where

$$\psi(x)=\sum_{n=1}^{\infty}\mathrm{e}^{-\pi n^2 x}.$$

If a time is allowed, an interlacing property for their zeros, an explicit formula, a three term recurrence relation and a combinatorial interpretation for the polynomials $p_n^{(m,\alpha,\beta)}(s)$ will be respectively introduced.

Row Sums and Alternating Sums of Riordan arrays

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(Joint work with Louis Shapiro)

Here we use their row sum generating functions and alternating sum generating functions to characterize Riordan arrays and subgroups of the Riordan group. A numerous applications and examples are presented which include the construction of the Girard-Waring type identities. We also show several extensions to weighted sum (generating) functions and to the sum functions for some non-Riordan arrays, which give a different view for considering Bernoulli and Euler polynomial sequences. Finally, we use sum (generating) functions to derive conjugate Bernoulli and conjugate Euler polynomial sequences.

Multivariate Riordan bases

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The method of generating functions is enhanced by differentials. To a sequence of numbers with combinatorial interests, we associate it not a function, neither a power series, but a differential. There is a pairing given by local cohomology residues for differentials and systems of parameters. The pairing is an algebraic analogue of integrations of differential forms in geometry. It has an effect of equating coefficients in a way independent of choices of a set of variables. Interplay of different sets of variables interprets many important theorems in combinatorial analysis. Useful formulas such as Lagrange inversions are build into our framework.

The method of generating differentials is based on the fact that a power series ring can be defined not referring to variables. To represent a power series in one variable in a more flexible way, we have introduced Schauder bases. Arbitrary Schauder bases may be not amenable to algebraic manipulations. A Riordan basis given by a proper Riordan array is a special type of Schauder basis, for which local cohomology residues work well. Many applications of Riordan arrays can be obtained by computations using Riordan bases. The notion of Schauder bases carries over effortlessly to power series in several variables. In the talk, we will define Riordan bases in several variables and demonstrate how local cohomology residues work for specific problems.

We give a simple example to sketch our approach. Let

$$CdX := \left(\sum C_i X^i\right) dX$$

be the differential associated to the Catalan numbers C_n characterized by the recurrence relation $C = 1 + XC^2$. For $n > 0$, the pairing for the differential CdX (called the numerator) and the system of parameter X^{n+1} (called the denominator) given by

$$C_n = \operatorname{res}\begin{bmatrix} CdX \\ X^{n+1} \end{bmatrix} = \frac{1}{n}\operatorname{res}\begin{bmatrix} dC \\ X^n \end{bmatrix}$$

is an analogue of a contour integration followed by the division by the complex number $2\pi\sqrt{-1}$. In the power series ring $\mathbb{Q}[[X]]$, there is another variable $Y := C - 1$. In other words, we have $\mathbb{Q}[[X]] = \mathbb{Q}[[Y]]$ with the relation $X = Y/(1+Y)^2$. The equality

$$\begin{bmatrix} dY \\ X^n \end{bmatrix} = \begin{bmatrix} (1+Y)^{2n}dY \\ Y^n \end{bmatrix}$$

is obtained by multiplying the numerator dY and the denominator X^n of the generalized fraction on the left-hand side by $(1+Y)^{2n}$. Since $dY = dC$, the binomial theorem gives

$$C_n = \frac{1}{n}\operatorname{res}\begin{bmatrix} (1+Y)^{2n}dY \\ Y^n \end{bmatrix} = \frac{1}{n}\binom{2n}{n-1}.$$

Here lies the central theorem of the method of generating differentials: The residue map is independent of choices of a set of variables.

Multidimensional Riordan arrays

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(Joint work with Gi-Sang Cheon)

For the set of nonnegative integers $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$, let $\mathbf{N}_d = \prod_{j=1}^{d} \mathbb{N}_0$ for $d \geq 2$. A d-dimensional infinity matrix $M = [M(\mathbf{x}_d) | \mathbf{x}_d \in \mathbf{N}_d]$ over the complex field $\mathbb{C}$ is defined as a map

$$M : \mathbf{N}_d \to \mathbb{C}.$$

We can generalize rows and columns of a matrix as lines of a multidimensional matrix. A line of M is a set of positions $(x_1, x_2, \ldots, x_d)$ obtained by fixing $d-1$ of $x_1, x_2, \ldots, x_d$ and allowing the other index to vary in $\mathbb{N}_0$.

In this talk, we introduce the concept of multidimensional Riordan arrays. The lines of these arrays are defined by a finite sequence of generating functions over the ring of formal power series $\mathbb{C}[[z]]$. If the sequence is of two generating functions, the array is Riordan. We prove that the set of multidimensional Riordan arrays forms a group. Particularly, we focus on 3-dimensional Riordan arrays $D = [D(n, k, \ell)]$ defined as

$$D(n, k, \ell) = [z^n] g f^k h^\ell$$

where $g(0) \neq 0, f(0) = 0, h(0) \neq 0$ and $f'(0) \neq 0$. We then discuss a group extension problem of the Riordan group and their combinatorial interpretations.

On the multiple orthogonal polynomials of Sheffer type

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(Joint work with Gi-Sang Cheon)

Let $\mathcal{P}$ be a vector space of polynomials over the complex field $\mathbb{C}$ and let $\mathcal{P}^*$ be its algebraic dual i.e., the set of all linear functionals $u : \mathcal{P} \to \mathbb{C}$. We denote $u(y)$ for $y \in \mathcal{P}$ by $\langle u, y \rangle$. Let $\{P_n\}_{n\geq 0}$ be a sequence of monic polynomials $P_n \in \mathcal{P}$ of degree n. For a positive integer d, the sequence $\{P_n\}_{n\geq 0} \subset \mathcal{P}$ is d-orthogonal if there exists a d-dimensional vector of linear functionals, $\mathcal{U} = (u_0, \ldots, u_{d-1})^T$, such that

$$\begin{cases} \langle u_k, P_r P_n \rangle = 0 & \text{if } r > nd + k, \quad n \in \mathbb{N} = \{0, 1, 2, \ldots\} \\ \langle u_k, P_n P_{nd+k} \rangle \neq 0 & \text{if } n \in \mathbb{N} \end{cases} \tag{1}$$

for each integer $k \in \{0, 1, \ldots, d-1\}$. A d-orthogonal polynomial sequence $\{P_n\}_{n\geq 0}$ is called of Sheffer type for (g, f) if there exist a pair of functions $g(z)$ and $f(z)$ where $g(0) \neq 0$, $f(0) = 0$ and $f'(0) \neq 0$ such that

$$g(t)e^{xf(t)} = \sum_{n\geq 0} S_n(x) \frac{z^n}{n!}.$$

In this talk, we consider d-orthogonal polynomial sequences $\{P_n(x)\}_{n\geq 0}$ of Sheffer type. Specifically, we discuss that

(i) a necessary and sufficient condition to be a Seffer type;
(ii) a d-dimensional vector of linear functionals for $\{P_n(x)\}_{n\geq 0}$;
(iii) the zeros of $\{P_n(x)\}_{n\geq 0}$;
(iv) the equivalence relation on the set of d-orthogonal polynomials of Seffer type.

The Riordan Group in Several Variables

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(Joint work with Gi-Sang Cheon)

Let $z_1, \ldots, z_d$ be indeterminates and $\mathbb{C}[\![z_1, \ldots, z_d]\!]$ the ring of d-variate formal expressions of the form $\sum_{\mathbf{i}} a_{\mathbf{i}} \mathbf{z}^{\mathbf{i}}$ where $a_{\mathbf{i}} = a_{i_1,\ldots,i_d}$ and $\mathbf{z}^{\mathbf{i}}$ denotes the monomial $z_1^{i_1} \cdots z_d^{i_d}$ for $i_j \in \mathbb{N}_0$. Let G be a d-variate generating function and $\mathbf{F} = (F_1, \ldots, F_d)$ be a vector function of d generating functions in any number of variables, all with vanishing constant terms. Define the formal composition by $G \circ \mathbf{F} = G(F_1, \ldots, F_d)$.

In this talk, we consider the set $\{(G, \mathbf{F})\}$ where $G(\mathbf{0}) \neq 0$ and the Jacobian determinant of $\mathbf{F}$ is not zero i.e., $\det\left(\frac{\partial F_j}{\partial z_i}\right) \neq 0$. Then we show that the set $\{(G, \mathbf{F})\}$ forms a group under the binary operation

$$(G, \mathbf{F})\,(H, \mathbf{L}) = (G(H \circ \mathbf{F}), \mathbf{L} \circ \mathbf{F}).$$

Furthermore, we give a matrix representation for the group element $(G, \mathbf{F})$ which will be called a d-Riordan matrix. All its entries might be expressed in terms of the form

$$\left[z_1^{i_1} \cdots z_d^{i_d}\right] G\,(\mathbf{z})\, F_1^{j_1}\,(\mathbf{z}) \cdots F_d^{j_d}\,(\mathbf{z}),$$

and the d-Riordan matrix is not necessarily lower triangular.

Construction of involutions in the Riordan group

Ana Luzón

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(Joint work with Manuel A. Morón and Felipe Prieto)

Knowing the involutions of a group is an important part to know the structure of that group. Shapiro in [8] asked about involutions in the Riordan group. Since then, some related works appeared in the literature. Up to my knowledge [1],[2],[3],[4],[6],[7]. In this talk I am going to show how to construct all involutions in each Riordan group of finite matrices. Then, by means of an inverse limit approach to the Riordan group, [5], we get all infinite ones. I will present some new subgroups and properties.

References

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q-cones: A toy example on Combinatorics and Topology of simplicial complexes

Manuel Alonso Morón

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(Joint work with Ana Luzón and Felipe Prieto)

One on the main tool to study the combinatorics of simplicial complexes is, the so called, the f-vector of such complexes which contains all information about the number and the dimension of the faces in the complexes.

In this talk we want only to point out that certain iterative constructions on Geometry and Topology, the so called joins of complexes, can be codify by means of very simple Riordan matrices, if we start at suitable initial conditions. So, we obtain that the Riordan pattern is not only in counting faces but also in computing the reduced Betti numbers of the corresponding polyhedra.

We want also to show a combinatorial property for q-cones which is analogous to the Euler characteristic and whose invariance in the appropriate framework is completely proved by Riordan matrices methods.

Along the talk and if the time permits, we will comment how Riordan matrices are involved in many results in the Combinatorics (and Topology) of simplicial complexes. For example, how to pass from f-vectors to the so called h-vectors, the re-interpretation of the Dehn-Sommerville equations, etc.

Patterns in Riordan arrays

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(Joint work with Donatella Merlini)

In this talk we examine some Riordan arrays by using modular arithmetics. One nice thing about modular arithmetic is that there are only a finite number of possible answers. If we assign to each of the possible answers a color, then the triangle can be presented as an array of colored dots or circles and in many cases we can find very interesting patterns in the corresponding image. This is well known in the case of Pascal triangle but we can illustrate many other Riordan arrays which give rise to other interesting situations which are worth to be investigated also from the algebraic and combinatorial point of view. We present some results concerning arrays related to Catalan, Motzkin and Fibonacci numbers or related to the enumeration of binary strings avoiding a pattern as well as to the corresponding inverse arrays.

The derived series of the Riordan group

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(Joint work with Ana Luzón and Manuel A. Morón)

In this talk, we will describe the derived series of the Riordan group. We will show that the Ore property holds, that is, any element in the n-th derivative subgroup of the Riordan group is a commutator of elements in the $(n-1)$-derivative subgroup of the Riordan group.

To do this, the structure of inverse limit in the Riordan group developed in [1] will be needed.

In particular, finding the second derivative subgroup answers an open question proposed by L. Shapiro in 2001 in [2].

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Krylov matrices versus orthogonal polynomials

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(Joint work with Gi-Sang Cheon)

Let $(p_n(x))_{n\geq 0}$ be a sequence of orthogonal polynomials p_n of degree n satisfying the three term recurrence relation:

$$p_{n+1}(x) = (x - a_n)p_n(x) - b_n p_{n-1}(x)\,.$$

The Hankel determinant associated to the sequence $(p_n(x))_{n\geq 0}$ is known as

$$\det H = b_1^n b_2^{n-1} \cdots b_n^1 \tag{2}$$

for any sequence (a_n).

In this talk, we are interested in an $n \times n$ matrix whose determinant can be expressed as the form (2). We show that if A is a lower triangular matrix whose diagonal entries are the same then the Krylov matrix defined as $K(A, \mathbf{v}) = \begin{bmatrix} \mathbf{v} & A\mathbf{v} & A^2\mathbf{v} & \cdots & A^{n-1}\mathbf{v} \end{bmatrix}$ has

$$\det(K(A, \mathbf{v})) = v_1^n a_{2,1}^{n-1} a_{3,2}^{n-2} \cdots a_{n,n-1}^1 \tag{3}$$

for any vector $\mathbf{v} \in \mathbb{R}^n$. For example, if A is the Pascal matrix, a Riordan matrix of the form $\left(\frac{1}{1-z}, \frac{z}{1-z}\right)$, and $\mathbf{v} = (1, 1, \ldots, 1)^T$, then $K(A, \mathbf{v})$ is the Vandermonde matrix with the determinant $1^n 1^{n-1} 2^{n-2} \cdots (n-1)^1$. This is motivated from the geometric progression matrices. In addition, we discuss how Krylov matrix $K(A, \mathbf{v})$ with the determinant (3) can be contributed to the orthogonal polynomials.

q-Analogue of Riordan Representation

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In [1], Shapiro and et al. formed a group which is called Riordan group. In [2], the authors proved that q-Riordan matrix can be represented by aid of the Eulerian generating functions and they defined q-Riordan group. In this study, using new binary operations, denoted by $*_q$ and $*_{1/q}$, and special q-operators, we obtain $q-$analogue of Riordan representation. Also we show that any q-matrices can be written as a pair of q-Riordan by aid of this representation. Specially, we get q-analogue of Riordan representation of q-Pascal matrix and inverse matrix.

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