ON THE NUMBER OF HILLS AMONG GENERALIZED DYCK PATHS

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It is well known that Dyck paths without hills (that is, ground level subpaths of the form \( UD \)) are counted by the Fine numbers, with generating function \([2]\)

\[
F(z) = \frac{1 - \sqrt{1 - 4z}}{3z - z\sqrt{1 - 4z}}.
\]

The main object of study in this paper is a generalization of the Fine number sequence. We let \( F_t(z) \) denote the generating function for the number of generalized Dyck paths without hills (that is, minimal length subpaths of the form \( U^{t-1}D \)). When \( t = 2 \), \( F_2(z) \) corresponds to the Fine number sequence and when \( t = 3 \), \( F_3(z) \) gives the number of ternary paths with no hills; that is, paths from \((0,0)\) to \((3n,0)\), using steps of the form \( U = (1,1) \) and \( D = (1,-2) \), never going below the \( x \)-axis and having no hills.

We consider a question posed by L. Shapiro circa 2009: “What is the asymptotic proportion of Dyck paths having an even number of hills?” We provide an answer to this question for generalized Dyck paths, finding in the case when \( t = 3 \) that the probability of a ternary path having an even number of hills approaches \( 125/169 \) as path length approaches infinity. We give a limiting distribution for the number of hills among generalized Dyck paths. We also consider a generalization of the following identity:

\[
(2z + z^2)F^2(z) - (1 + 2z)F(z) + 1 = 0,
\]

for which a combinatorial proof was provided in [1]. We generalize the result to \( F_t(z) \) in the following way

\[
1 + zF_t'(z) = \sum_{k=0}^{t-1} \binom{t-1}{k} + \binom{t}{k+1}z (-1)^k z^k F_{k+1}^t(z)
\]
and explore various combinatorial interpretations of the results.

REFERENCES