On the number of hills among generalized Dyck paths

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It is well known that Dyck paths without hills (that is, ground level subpaths of the form UD) are counted by the Fine numbers, with generating function [2]

$$F(z) = \frac{1-\sqrt{1-4z}}{3z - z\sqrt{1-4z}}\,.$$

The main object of study in this paper is a generalization of the Fine number sequence. We let $F_t(z)$ denote the generating function for the number of generalized Dyck paths without hills (that is, minimal length subpaths of the form $U^{t-1}D$). When $t=2$, $F_t(z)$ corresponds to the Fine number sequence and when $t=3$, $F_t(z)$ gives the number of ternary paths with no hills; that is, paths from $(0,0)$ to $(3n,0)$, using steps of the form $U=(1,1)$ and $D=(1,-2)$, never going below the x-axis and having no hills.

We consider a question posed by L. Shapiro circa 2009: "What is the asymptotic proportion of Dyck paths having an even number of hills?" We provide an answer to this question for generalized Dyck paths, finding in the case when $t=3$ that the probability of a ternary path having an even number of hills approaches 125/169 as path length approaches infinity. We give a limiting distribution for the number of hills among generalized Dyck paths. We also consider a generalization of the following identity:

$$(2z+z^2)F^2(z) - (1+2z)F(z) + 1 = 0\,,$$

for which a combinatorial proof was provided in [1]. We generalize the result to $F_t(z)$ in the following way

$$1 + zF_t^t(z) = \sum_{k=0}^{t-1} \left[\binom{t-1}{k} + \binom{t}{k+1} z \right] (-1)^k z^k F_t^{k+1}(z)$$

and explore various combinatorial interpretations of the results.

References

[1] G. Cheon, S. Lee, L. Shapiro, *The Fine numbers refined*, European J. Combin. **31** (2010), 120–128.

[2] E. Deutsch, L. Shapiro, *A survey of the Fine numbers*, Discrete Mathematics **241** (2001), 241–265.