The Riemann Zeta function and associated polynomials

Gi-Sang Cheon

Department of Mathematics
Sungkyunkwan University
Suwon 440-746, Republic of Korea

(Joint work with Hana Kim)

The Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

is defined for complex numbers s with $\mathrm{Re}(s) > 1$. The Riemann hypothesis asserts that all nontrivial zeros of $\zeta(s)$ lie on the critical line $\sigma = \frac{1}{2}$. Riemann also extended $\zeta(s)$ to all complex numbers $s \neq 1$ by deriving the analytic continuation using the theta series

$$\theta(x) = \sum_{n=-\infty}^{\infty} \mathrm{e}^{-\pi n^2 x}.$$

In this talk, we introduce a family of polynomials $p_n^{(m,\alpha,\beta)}(s)$ for real numbers m, α, β that satisfies the analog of the Riemann hypothesis. Specifically, we show that all the zeros of $p_n^{(m,\alpha,\beta)}(s)$ lie on the line $\mathrm{Re}(s) = \frac{m}{2}$. Further, using Riordan array technique we show that the polynomials $p_n^{(m,\alpha,\beta)}$ lead to a generalization of the Riemann's integral

$$\int_0^{\infty} \psi(x) x^{\frac{s}{2}-1}\, \mathrm{d}x$$

arising in the Riemann's second proof of the analytic continuation of $\zeta(s)$ where

$$\psi(x) = \sum_{n=1}^{\infty} \mathrm{e}^{-\pi n^2 x}.$$

If a time is allowed, an interlacing property for their zeros, an explicit formula, a three term recurrence relation and a combinatorial interpretation for the polynomials $p_n^{(m,\alpha,\beta)}(s)$ will be respectively introduced.