Multivariate Riordan bases

Huang, I-Chiau

Institute of Mathematics
Academia Sinica
Taipei, Taiwan

The method of generating functions is enhanced by differentials. To a sequence of numbers with combinatorial interests, we associate it not a function, neither a power series, but a differential. There is a pairing given by local cohomology residues for differentials and systems of parameters. The pairing is an algebraic analogue of integrations of differential forms in geometry. It has an effect of equating coefficients in a way independent of choices of a set of variables. Interplay of different sets of variables interprets many important theorems in combinatorial analysis. Useful formulas such as Lagrange inversions are build into our framework.

The method of generating differentials is based on the fact that a power series ring can be defined not referring to variables. To represent a power series in one variable in a more flexible way, we have introduced Schauder bases. Arbitrary Schauder bases may be not amenable to algebraic manipulations. A Riordan basis given by a proper Riordan array is a special type of Schauder basis, for which local cohomology residues work well. Many applications of Riordan arrays can be obtained by computations using Riordan bases. The notion of Schauder bases carries over effortlessly to power series in several variables. In the talk, we will define Riordan bases in several variables and demonstrate how local cohomology residues work for specific problems.

We give a simple example to sketch our approach. Let

$$CdX := \left(\sum C_i X^i\right) dX$$

be the differential associated to the Catalan numbers C_n characterized by the recurrence relation $C = 1 + XC^2$. For $n > 0$, the pairing for the differential CdX (called the numerator) and the system of parameter X^{n+1} (called the denominator) given by

$$C_n = \operatorname{res}\begin{bmatrix} CdX \\ X^{n+1} \end{bmatrix} = \frac{1}{n}\operatorname{res}\begin{bmatrix} dC \\ X^n \end{bmatrix}$$

is an analogue of a contour integration followed by the division by the complex number $2\pi\sqrt{-1}$. In the power series ring $\mathbb{Q}[[X]]$, there is another variable $Y := C-1$. In other words, we have $\mathbb{Q}[[X]] = \mathbb{Q}[[Y]]$ with the relation $X = Y/(1+Y)^2$. The equality

$$\begin{bmatrix} dY \\ X^n \end{bmatrix} = \begin{bmatrix} (1+Y)^{2n}dY \\ Y^n \end{bmatrix}$$

is obtained by multiplying the numerator dY and the denominator X^n of the generalized fraction on the left-hand side by $(1+Y)^{2n}$. Since $dY = dC$, the binomial theorem gives

$$C_n = \frac{1}{n}\operatorname{res}\begin{bmatrix} (1+Y)^{2n}dY \\ Y^n \end{bmatrix} = \frac{1}{n}\binom{2n}{n-1}.$$

Here lies the central theorem of the method of generating differentials: The residue map is independent of choices of a set of variables.