Characterization and history of Riordan arrays

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My talk will be divided into two parts, one concerning the diagonals top-left to bottom-right of Riordan arrays, and the other a short history of these matrices. It is important to realize that the results obtained can be extended to recursive matrices, which seems the most natural setting where these properties should be studied.

Riordan arrays have been studied from many points of view, corresponding to different characterizations and construction methods. Their classical definition is $D = \mathcal{R}(d(t),\ h(t))$, where $d(t)$ (the *boundary value*) is the generating function of column 0, and $h(t)$ (the *recurrence rule*) allows to pass from any column to the successive. This definition privileges an approach based on columns, while if one prefers a row based study, the A-sequence reveals to be more appropriate. Unfortunately, at least to my knowledge, an approach through diagonals has never been tried.

Some properties of Riordan arrays can be easily observed by this approach. For example, diagonal 0 is a geometric progression, often having ratio $q = 1$ as in the Pascal triangle. Analogously, the diagonal 1 contains an arithmetic-geometric progression, but as we pass to the other diagonals, things become more complex and their structure is difficult to analyze.

In this part of my talk, I'll discuss some results obtained on diagonals of Riordan arrays. Several other points remain at a conjecture level, and I thank in advance every one, who will take a part in this research with hints or suggestions of every kind.

The vast literature on Riordan arrays, piled up in few years, evidences that these matrices cover important sectors of Mathematics (Combinatorial Analysis, but also parts of Algebra and Geometry), are easily handled and furnish effective results. In particular, the subset of all Riordan arrays having $d(0) \neq 0$, $h(0) = 0$ and $h'(0) \neq 0$, constitutes a group, the *Riordan Group* of *proper* Riordan arrays, having as operation the usual row-by-column product, in this context, however, corresponding to simple manipulations of algebraic formulae.

These concepts were formulated at the beginning of the last decade of the second millennium, but the authors were well aware that analogous ideas had been discussed starting 130 years earlier. In fact, the French Mathematician Blissard introduced what he called *Umbral Calculus*, a formalism that shadows (the Latin *Umbra* is *Shadow* in English) some algebraically complex constructs with simpler ones. This calculus was used extensively by Lucas, Bell and eventually by G.-C. Rota and S. Roman, who gave a modern and rigorous foundation to the formalism.

In the formulation of Rota-Roman, the Riordan group is just the 1-Umbral Calculus, but this latter approach did not receive so much attention by researchers. Why? I'll dedicate the last part of my talk to explain the reasons of this strange fact, and some of them can be listed in this Abstract: the notation different from the one currently used in Combinatorics; the apparent limitation of applications restricted to orthogonal polynomials; the emphasis given to the exponential case; and others, less relevant.